

Multiscale seamless-domain method for solving nonlinear heat conduction problems without iterative multiscale calculations

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Abstract

In this paper, we applied a multiscale numerical scheme called the seamless-domain method (SDM) to nonlinear elliptic boundary value problems. Although the SDM is meshfree, it can obtain a high-resolution solution whose dependent-variable gradient(s) is sufficiently smooth and continuous. The SDM models with only coarse-grained points can produce accurate solutions for both linear heat conduction problems and linear elastic problems. This manuscript presents a simple nonlinear solver for the SDM analysis of heterogeneous materials. Although the solver can easily approximate the solutions to nonlinear multiscale problems, it does not require an iterative multiscale analysis at every convergence calculation. In other words, the proposed scheme does not completely interactively couple the multiple scales. We present numerical examples of nonlinear stationary heat conduction analyses of heterogeneous fields and compare the SDM model, the direct finite-element model, and the homogenized model based on the homogenization theory. For a real heterogeneous structure (graphite fiber composite) that did not have strong material nonlinearities, the SDM model using only 925 points gave a solution with similar precisions as an ordinary finite element solution using hundreds of thousands of nodes. To investigate the limitations of the method, we also applied the SDM to imaginary materials with various strengths of thermal property nonlinearities.

Key words : Composite material, Numerical analysis, Nonlinear problem, Finite element method, Heat conduction, Multiscale solver, Meshfree method, Homogenization

1. Introduction

In previous work (Suzuki and Soga, 2016, Suzuki, 2016a, 2016b), a multiscale analysis called the seamless-domain method (SDM) was developed and applied to linear analyses. When conducting a macroscopic analysis of an entire field, the SDM constructs a meshfree model that is represented by only a small number of coarse-grained points (CPs). The entire field has no elements, mesh, cell, or grid. Even if the field is heterogeneous and composed of multiple constituents, the solution can express the complicated behaviors without homogenization.

There have been several notable studies related to multiscale modeling techniques such as multiscale finite-element methods (FEMs) (Hesthaven, et al., 2014; Ilic and Hackl, 2009; Abdulle and Engquist, 2007), multiscale finite-volume methods (Jenny, et al., 2003 and 2004, Lunati and Jenny, 2006), homogenization methods (Terada, et al., 2000; Kaczmarczyk, et al., 2010), meshfree multiscale techniques (Liu, et al., 2000; Zhang, et al., 2012), and mesh superposition techniques for the FEM (Fish, et al., 1994; Park, et al., 2003; Jiang, et al., 2008; Rojek and Oñate, 2007). However, they are formulated in a different way to the SDM. Compared with conventional monoscale and multiscale schemes, the advantages and differences of the SDM are as follows.

- When modeling an entire global simulated domain (an entire structure), the SDM requires only points and does not require a stiffness equation, mesh, element, cell, or grid.
- The SDM can generate an almost continuous dependent-variable distribution with continuous gradient(s) throughout the simulated field except on interfaces between different materials having different material properties.

- Even if an analytical field is heterogeneous and has multiple components, there is no need to construct individual constituent models. However, the interpolating functions of the SDM can reproduce a high-resolution variable contour, which considers the locations of the structural heterogeneity.
- The special interpolation functions of the SDM can be obtained without a moving least-squares scheme, polynomial functions, or radial basis functions.
- The SDM does not suffer from error due to the size effect. It is applicable regardless of the scale of the ratio of a unit cell to the entire field (Suzuki and Soga, 2016).

In a previous study (Suzuki and Soga, 2016), the practicality of the SDM scheme was investigated for example linear problems of stationary temperature fields. The accuracy of the SDM model represented by only a few hundred points was equivalent to a conventional FEM with tens of thousands of nodes, when analyzing both a homogeneous domain and heterogeneous field. The temperature and its gradient calculated by the proposed method were continuous everywhere in the material. Another study (Suzuki, 2016a) gave numerical examples that applied the method to linear elastic fields. The SDM with a few hundred points generated shear-locking-free and hourglass-free solutions at the same level of analytical precision as solutions computed by the ordinary FEM using hundreds of thousands of nodes. However, previous studies have only conducted the SDM analysis for linear problems, and it has not been applied to a nonlinear problem.

The present manuscript presents a simple and quick nonlinear solver based on the SDM theory for elliptic boundary value problems. The nonlinear technique is used to solve numerical examples of nonlinear heat conduction fields, where the material's thermal conductivity changes according to its temperature. The target structures are an actual composite material (graphite fiber/epoxy polymer composite) and imaginary heterogeneous materials with significantly larger material nonlinearities. We compared the accuracy of the SDM with that of the ordinary FEM (Section 3). Although the target structures in the examples are several hundred micrometers in length, actual fiber composite components and products are much larger than the structures. Hence, we illustrate how to apply the proposed SDM scheme to real-size composite components in Appendix.

In general, standard multiscale analyses of nonlinear problems require iterative multiscale calculations that exactly couple microscopic and macroscopic analyses (Roters, et al., 2010). Microscopic and macroscopic nonlinear analyses are repeatedly conducted until the microscopic solution converges to the macroscopic solution. Although this has a significant impact on the computational cost, it is the most reliable and accurate nonlinear solver. Many nonlinear solvers have been developed for multiscale modeling and successfully applied to simulate elasto-plastic deformation tendency (Jiang, et al., 2008; McDowell, 2010) and to predict structural failure behavior of composite materials (Fish and Yu, 2001). However, the above exact techniques are very time-consuming and might be unrealistic in some cases. Although parallel computing of the microscopic analysis would somewhat increase the computational efficiency (Markovic, et al., 2005), a large multiscale model (e.g., an aircraft component made of fiber composite material) would still be extremely expensive to analyze.

Similar to the linear techniques (illustrated in (Suzuki and Soga, 2016; Suzuki, 2016a, 2016b) and Subsection 2.3), the nonlinear solver of the SDM is composed of two analyses. The first is a microscopic analysis of the local and small domain, to construct special functions which interpolate dependent-variable distribution in the local domain. The functions capture the boundary effects, singularities, and microscopic features of a heterogeneous structure. This means that an SDM solution can represent a heterogeneous material with microscopic constituents, without homogenization. The second analysis considers the entire global domain using the interpolation functions.

In the previous version of the SDM (linear solver (Suzuki and Soga, 2016; Suzuki, 2016a, 2016b)), the interpolating functions do not depend on the dependent variable of the simulated field. However, in the nonlinear solver of the SDM, the functions change according to the dependent variable. The first step of the presented scheme conducts a microscopic analysis several times, to construct various kinds of interpolation functions for various dependent-variable values. In other words, the interpolating functions are derived based on the nonlinear behaviors of the field. By preparing these functions before the macroscopic analysis, we can eliminate exact couplings between the macroscopic and microscopic analyses. Then, we can approximately solve nonlinear problems using the interpolation functions without any subsequent microscopic analysis. Our nonlinear solver is thus computationally inexpensive. Details of the procedure for the proposed nonlinear technique is given in Section 2.4.

Note that this study is only an attempt at approximately solving nonlinear problems. Although the method could be applied to problems with relatively weak material nonlinearities, there is no guarantee that it would provide an exact

solution for any problem with a strong nonlinearity. To investigate the limitations of our method, we used the nonlinear solver to analyze imaginary materials with various strengths of material nonlinearities. Our technique can be conveniently implemented and applied to general problems governed by nonlinear elliptic partial differential equation(s), such a nonlinear Laplace's equation.

2. Procedure of the SDM

2.1 Problem statement

Hereafter, we consider a two-dimensional steady-state temperature problem to illustrate the computational procedure of the SDM. The dependent variable, its gradient, and flux are the temperature, temperature gradient, and heat flux, respectively. Target simulated field is called a global domain (Ω_G in Fig. 1(a)) in this article. Ω_G is a scalar temperature field whose boundary Γ_G is a smooth manifold. Ω_G is a heterogeneous structure and assumed to have a periodic microstructure, called a unit cell. This has a circular inclusion in the base material, as shown in Fig. 1(a).

Note that actual fiber composite components and products are much larger than Ω_G in Fig. 1(a). Practical application of the proposed SDM technique to a real-size composite having enormous number of inclusions are excluded here because they are illustrated in Appendix.

Unless heat enters from outside the system, Ω_G is governed by the following Laplace's equation.

$$D_{11} \frac{\partial^2 u}{\partial x_1^2} + D_{22} \frac{\partial^2 u}{\partial x_2^2} = 0 \quad \text{for arbitrary } \mathbf{x} = [x_1 \ x_2]^T \in \Omega_G \quad (1)$$

where $\mathbf{x} = [x_1 \ x_2]^T$ is the position vector, $u = u(\mathbf{x})$ is the temperature at $\mathbf{x}$, and D_{11}, D_{22} are the thermal conductivities in the x_1 and x_2 directions. The superscript "T" refers to the transposition of a matrix or vector. Γ_G is composed of the Dirichlet boundary (Γ_u) and the Neumann boundary (Γ_q).

$$\Gamma_u \cup \Gamma_q = \Gamma_G,$$

$$\Gamma_u \cap \Gamma_q = \phi.$$

where ϕ is an empty set. u is fixed on Γ_u and the temperature gradient in the direction normal to Γ_q is fixed.

When both D_{11} and D_{22} in Eq. (1) are constant regardless of the temperature (u),

$$\begin{aligned} D_{11} &= D_{11}(\mathbf{x}) \quad \text{for arbitrary } \mathbf{x} \in \Omega_G \\ D_{22} &= D_{22}(\mathbf{x}) \end{aligned} \quad (2)$$

Equation (1) is a linear heat conduction equation and can be accurately solved using the previous version of the SDM (linear solver) presented in Subsection 2.3 and (Suzuki and Soga, 2016).

If either D_{11}, D_{22} , or both change according to u , then

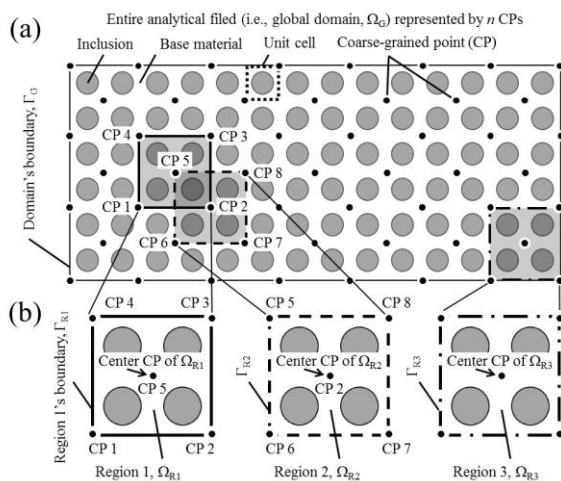


Fig. 1 Example of the heterogeneous global domain having a periodic microstructure (unit cell) in (a); regions of influence that are partially extracted from the global domain in (b).

$$\begin{aligned} D_{11} &= D_{11}(\mathbf{x}, u) \quad \text{for arbitrary } \mathbf{x} \in \Omega_G \\ D_{22} &= D_{22}(\mathbf{x}, u) \end{aligned} \quad (3)$$

Now, Eq. (1) becomes a nonlinear elliptic partial differential equation and cannot be solved by the linear SDM solver. We propose the simple SDM solver for the nonlinear problem that is illustrated in Subsection 2.4.

2.2 Outline of the calculation procedure of the SDM

This subsection illustrates outline of the calculation procedure of the SDM with the stationary temperature field governed by Eq. (1) as an example. The outline is the same irrespective of whether the governing equation of the field is linear or nonlinear.

As depicted in Fig. 1(a), the entire analytical field (i.e., global domain, Ω_G) is represented by n coarse-grained points (CPs) in the SDM. The CPs are endowed with dependent-variable values (temperature values in this case). The temperature vector on those points are denoted

$$\mathbf{u}^G = [u_1 \quad \cdots \quad u_n]^T \quad (4)$$

where u_i is temperature value at the i -th CP in Ω_G (i.e., CP i).

We divide Ω_G into n regions, $\Omega_{R1}, \Omega_{R2}, \dots, \Omega_{Rn}$, which are defined as regions of influence. Basically, sizes and shapes of the regions are arbitrary, but neighboring regions need to share some CPs and partially overlap each other, as depicted in Fig. 1(a). All regions in Fig. 1(b) are composed of two-by-two unit cells. It is noted that the number of the regions, n , is the same as the number of the CPs in Ω_G . Each region has a CP in the center and other surrounding CPs on the region's boundary.

For the i -th region (i.e., Region i , Ω_{Ri}), the temperature value at the center CP and those at the surrounding CPs are respectively denoted

$$u_{\text{explained}}^{Ri} \in \mathbf{R} \quad \text{for } i \in \{1, \dots, n\} \quad (5)$$

$$\mathbf{u}_{\text{predictor}}^{Ri} \in \mathbf{R}^{m_i} \quad \text{for } i \in \{1, \dots, n\} \quad (6)$$

Where $\mathbf{R}$ and m_i are a set of all real numbers and the number of the surrounding CPs on the boundary of Ω_{Ri} , respectively. As depicted in Fig. 1(b), for Region 1 (Ω_{R1}),

$$u_{\text{explained}}^{R1} = u_5 \in \mathbf{R} \quad (7)$$

$$\mathbf{u}_{\text{predictor}}^{R1} = [u_1 \quad u_2 \quad u_3 \quad u_4]^T \in \mathbf{R}^4 \quad (8)$$

where the number of the surrounding CPs in Ω_{R1} is $m_1 = 4$. For Region 2 (Ω_{R2}),

$$u_{\text{explained}}^{R2} = u_2 \in \mathbf{R} \quad (9)$$

$$\mathbf{u}_{\text{predictor}}^{R2} = [u_6 \quad u_7 \quad u_8 \quad u_5]^T \in \mathbf{R}^4 \quad (10)$$

where the number of the neighboring CPs in Ω_{R2} is $m_2 = 4$. Note that $\mathbf{u}_{\text{predictor}}^{Ri}$ does not include $u_{\text{explained}}^{Ri}$ for any i .

We consider interpolation functions that can estimate the temperature distribution in Ω_{Ri} referring to temperatures at the m_i surrounding CPs on the boundary of Ω_{Ri} .

$$u(\mathbf{x}) = f^{Ri}(\mathbf{x}, \mathbf{u}_{\text{predictor}}^{Ri}) \in \mathbf{R} \quad \text{for } \mathbf{x} \in \Omega_{Ri} \quad \text{and } i \in \{1, \dots, n\} \quad (11)$$

where $u(\mathbf{x})$ and f^{Ri} are the temperature value at position $\mathbf{x}$ and the interpolation function for Ω_{Ri} , respectively. We illustrate the detailed procedure to derive f^{Ri} for linear problems in Subsection 2.3 and the derivation of f^{Ri} for nonlinear problems in Subsection 2.4.1.

When we have the accurate interpolation functions for all the regions ($f^{R1}, \dots, f^{Rn}$), both the temperature values at all CPs ($\mathbf{u}^G$ in Eq. (4)) and the temperature distribution in the entire global domain (Ω_G) can be determined by the following procedure. First, we need to determine $\mathbf{u}^G$. Substituting the position of the center CP in Ω_{Ri} into $\mathbf{x}$ in Eq. (11), it follows that

$$u_{\text{explained}}^{Ri} = u(\mathbf{x}_{\text{explained}}^{Ri}) = f^{Ri}(\mathbf{x}_{\text{explained}}^{Ri}, \mathbf{u}_{\text{predictor}}^{Ri}) \quad \text{for } i \in \{1, \dots, n\} \quad (12)$$

where $\mathbf{x}_{\text{explained}}^{Ri}$ is the position of the center CP in Ω_{Ri} . As Eq. (12) holds for any number $i \in \{1, \dots, n\}$, we can regard them as being simultaneous equations that have n equations with n unknowns ($u_{\text{explained}}^{R1}, \dots, u_{\text{explained}}^{Rn}$). By imposing some boundary conditions on Ω_G , we can solve Eq. (12) and determine all the unknown values, which indicates that we obtain the temperature values at all CPs in Ω_G , $\mathbf{u}^G$.

When calculating $\mathbf{u}^G$, the global domain is represented by only a small number of CPs and does not require a node, mesh, grid, cell, element, or stiffness equation that determines the relation between the nodal temperature and nodal heat flow rate. Even if the global domain is heterogeneous, there is no necessity to separately model the microscopic constituents.

After determining $\mathbf{u}^G$, we reproduce the temperature distribution in the global domain. As shown in Eq. (11), the temperature profile in Region i , Ω_{Ri} , is interpolated from $\mathbf{u}_{\text{predictor}}^{Ri}$ using f^{Ri} . As $\mathbf{u}^G$ includes all components of $\mathbf{u}_{\text{predictor}}^{Ri}$ for any $i \in \{1, \dots, n\}$, we can interpolate the temperature distributions in all the regions from $\mathbf{u}^G$. We connect all the distributions to each other to generate the temperature profile throughout the global domain.

As shown in Fig. 1(a), neighboring regions share some CPs and are partially superposed with each other. For instance, Ω_{R1} and Ω_{R2} shares two CPs (CPs 2 and 5). The SDM formulation guarantees that the temperature value at each shared CP in Ω_{R1} is equal to that at the CP located at the same position in Ω_{R2} . Consequently, the two shared CPs attempt to equalize the temperature profile of the overlapped region in Ω_{R1} and the distribution in Ω_{R2} . When Ω_{R1} and Ω_{R2} have a sufficient number of shared CPs, the two distributions in the superposed area match exactly. This achieves sufficiently continuous connection of the temperature and its gradient on the interface between the two regions.

By describing regions centered at each of the CPs, the global domain is filled with shared regions. As there is no region or CP belonging to an isolated (single) region, it is assured that the temperature and its gradient(s) would be continuous throughout the global domain (except on interfaces between different type materials having different thermal properties). The global domain then has a smooth temperature profile and becomes “seamless”. The objective of the SDM is to construct the “seamless” global domain at low analytical cost.

From the above, the SDM technique gives a sufficiently accurate solution when all the following requirements are satisfied.

- Accurate interpolation functions are obtained for all regions of influence
- Neighboring regions of influence share a sufficient number of CPs so that dependent variable and its gradient(s) are sufficiently continuous at the interface between the regions
- All regions of influence have a sufficient number of CPs so that precise dependent-variable profile in the regions can be interpolated referring to the variables at the CPs on the regions' boundaries

2.3 SDM for linear problems

Before illustrating the proposed nonlinear solver, we briefly review the SDM formulations for solving linear problems. This subsection illustrates how to construct the interpolation functions (f^{Ri} in Eq. (11)) for two-dimensional linear stationary temperature field governed by the following equation.

$$D_{11}(\mathbf{x}) \frac{\partial^2 u(\mathbf{x})}{\partial x_1^2} + D_{22}(\mathbf{x}) \frac{\partial^2 u(\mathbf{x})}{\partial x_2^2} = 0 \quad (13)$$

In the linear field, the thermal conductivity are constant regardless of the temperature.

When the problem is linear, the interpolation functions are also independent from the temperature. The function for Region i , f^{Ri} , is expressed as a linear combination of temperature values at surrounding CPs on the boundary of Ω_{Ri} , i.e., $\mathbf{u}_{\text{predictor}}^{Ri}$ in Eq. (6). That is,

$$u(\mathbf{x}) = f^{Ri}(\mathbf{x}, \mathbf{u}_{\text{predictor}}^{Ri}) = \mathbf{N}^{Ri}(\mathbf{x}) \mathbf{u}_{\text{predictor}}^{Ri} \quad \text{for } \mathbf{x} \in \Omega_{Ri} \quad \text{and any } i \in \{1, \dots, n\} \quad (14)$$

where

$$\mathbf{N}^{Ri}(\mathbf{x}) = [N_1^{Ri}(\mathbf{x}) \quad \dots \quad N_{m_i}^{Ri}(\mathbf{x})] \in \mathbf{R}^{1 \times m_i} \quad (15)$$

is a one-by- m_i interpolation function matrix for Ω_{Ri} . m_i is the number of the surrounding CPs in Ω_{Ri} . Note that $\mathbf{N}^{Ri}$ depends on only the position $\mathbf{x}$ and does not have a temperature dependency.

Substituting the location of the center CP in Ω_{Ri} (i.e., $\mathbf{x}_{\text{explained}}^{Ri}$) into $\mathbf{x}$ in Eq. (14), it follows that

$$u_{\text{explained}}^{Ri} = u(\mathbf{x}_{\text{explained}}^{Ri}) = \mathbf{N}^{Ri}(\mathbf{x}_{\text{explained}}^{Ri}) \mathbf{u}_{\text{predictor}}^{Ri} = \mathbf{a}^{Ri} \mathbf{u}_{\text{predictor}}^{Ri} \quad \text{for } i \in \{1, \dots, n\} \quad (16)$$

where

$$\mathbf{a}^{Ri} = \mathbf{N}^{Ri}(\mathbf{x}_{\text{explained}}^{Ri}) \quad \text{for } i \in \{1, \dots, n\} \quad (17)$$

is called an influence coefficient matrix for Ω_{Ri} . Equation (16) is equivalent to Eq. (12) and indicates that $\mathbf{a}^{Ri}$ determines temperature at the center CP, $u_{\text{explained}}^{Ri}$, referring to $\mathbf{u}_{\text{predictor}}^{Ri}$. When the problem is linear and the material property is not dependent on the temperature, all the components of $\mathbf{a}^{Ri}$ are also constant.

Figure 2(a) shows an example for four surrounding CPs (CPs 1–4) on Region 1's boundary, i.e., $m_1 = 4$. Temperature profile in Region 1, Ω_{R1} , is estimated using temperatures at the four CPs $\mathbf{u}_{\text{predictor}}^{R1} = [u_1 \quad u_2 \quad u_3 \quad u_4]^T$. Therefore, the interpolation function for Ω_{R1} , f^{R1} , and the influence coefficient matrix, $\mathbf{a}^{R1}$, are expressed in the following forms:

$$u(\mathbf{x}) = f^{R1}(\mathbf{x}, \mathbf{u}_{\text{predictor}}^{R1}) = \mathbf{N}^{R1}(\mathbf{x}) \mathbf{u}_{\text{predictor}}^{R1} \quad \text{for } \mathbf{x} \in \Omega_{R1} \quad (18)$$

$$u_{\text{explained}}^{R1} = \mathbf{a}^{R1} \mathbf{u}_{\text{predictor}}^{R1} \quad (19)$$

where

$$\mathbf{N}^{R1}(\mathbf{x}) = [N_1^{R1}(\mathbf{x}) \quad \dots \quad N_4^{R1}(\mathbf{x})] \in \mathbf{R}^{1 \times 4} \quad (20)$$

$$\mathbf{a}^{R1} = [N_1^{R1}(\mathbf{x}_{\text{explained}}^{R1}) \quad \dots \quad N_4^{R1}(\mathbf{x}_{\text{explained}}^{R1})] \in \mathbf{R}^{1 \times 4} \quad (21)$$

The detailed procedure for deriving $\mathbf{N}^{R1}$ and $\mathbf{a}^{R1}$ is as follows. $\mathbf{N}^{R1}$ and $\mathbf{a}^{R1}$ are obtained from the result of stationary heat conduction analysis of a local domain Ω_{L1} with boundary Γ_{L1} , as depicted in Fig. 2(b). Although

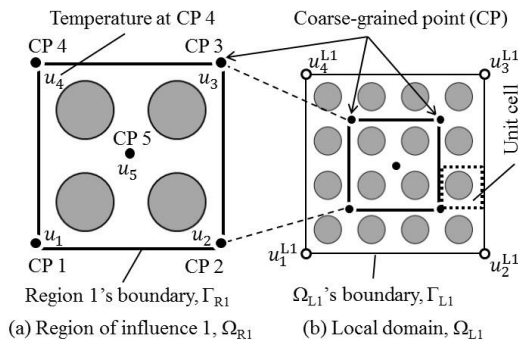


Fig. 2 Region of influence 1 (Ω_{R1}) that is the inner two-by-two unit cells in the local domain in (a) and Local domain 1 (Ω_{L1}) consisting of four-by-four unit cells in (b).

Ω_{L1} is composed of four-by-four unit cells, it need not necessarily be what is shown in Fig. 2(b). The internal two-by-two unit cells in Ω_{L1} are regarded as region of influence 1 (Ω_{R1} in Fig. 2). In this article, analysis of the local domain is conducted employing a standard linear finite-element solver but other conventional numerical scheme may be used.

First, the Ω_{L1} is modeled into a finite-element mesh where the constitutive materials are separately modeled. As shown in Fig. 2(b), four temperatures at four points denoted by $\mathbf{u}^{L1}$ are placed on Γ_{L1} :

$$\mathbf{u}^{L1} = [u_1^{L1} \quad u_2^{L1} \quad u_3^{L1} \quad u_4^{L1}]^T \in \mathbf{R}^4 \quad (22)$$

The number of components of $\mathbf{u}^{L1}$ need to be the same as the number of components of $\mathbf{u}_{\text{predictor}}^{R1}$. To find $\mathbf{N}^{R1}$, four kinds of fixed-temperature boundary conditions are placed on Γ_{L1} , as shown in Figs. 3(a)–(d):

$$\begin{aligned} \mathbf{u}^{L1} &= [1 \quad 0 \quad 0 \quad 0]^T, \mathbf{u}^{L1} = [0 \quad 1 \quad 0 \quad 0]^T, \\ \mathbf{u}^{L1} &= [0 \quad 0 \quad 1 \quad 0]^T, \mathbf{u}^{L1} = [0 \quad 0 \quad 0 \quad 1]^T \end{aligned} \quad (23)$$

Then, we analyze Ω_{L1} using the linear FEM solver. The outer frame of Ω_{L1} has a linear temperature profile that connects the temperatures at the corners of Ω_{L1} . When the boundary conditions of the first equation in Eq. (23) are imposed on Ω_{L1} , the temperature distribution in Ω_{R1} is given by Fig. 3(a). It has the form

$$u(\mathbf{x}) = A_1^{R1}(\mathbf{x}) \quad \text{for } \mathbf{x} \in \Omega_{R1} \quad (24)$$

Additionally, the temperatures at CPs 1–5 are

$$u_{\text{explained}}^{R1} = u_5 = A_1^{R1}(\mathbf{x}_{\text{explained}}^{R1}) \quad (25)$$

$$\mathbf{u}_{\text{predictor}}^{R1} = [A_1^{R1}(\mathbf{x}_{\text{CP1}}^{R1}) \quad A_1^{R1}(\mathbf{x}_{\text{CP2}}^{R1}) \quad A_1^{R1}(\mathbf{x}_{\text{CP3}}^{R1}) \quad A_1^{R1}(\mathbf{x}_{\text{CP4}}^{R1})]^T \quad (26)$$

where $\mathbf{x}_{\text{explained}}^{R1}$ and $\mathbf{x}_{\text{CP}i}^{R1}$ are the position of the center CP and the location of CP i in Ω_{R1} , respectively. Similarly, in Ω_{L1} with the i -th boundary conditions in Eq. (23), the temperatures in Ω_{R1} and at CPs 1–5 are:

$$\begin{aligned} u(\mathbf{x}) &= A_i^{R1}(\mathbf{x}) \quad \text{for } i \in \{1, \dots, 4\} \quad (27) \\ u_{\text{explained}}^{R1} &= u_5 = A_i^{R1}(\mathbf{x}_{\text{explained}}^{R1}) \\ \mathbf{u}_{\text{predictor}}^{R1} &= [u_1 \quad u_2 \quad u_3 \quad u_4]^T = [A_i^{R1}(\mathbf{x}_{\text{CP1}}^{R1}) \quad A_i^{R1}(\mathbf{x}_{\text{CP2}}^{R1}) \quad A_i^{R1}(\mathbf{x}_{\text{CP3}}^{R1}) \quad A_i^{R1}(\mathbf{x}_{\text{CP4}}^{R1})]^T \end{aligned}$$

Because Ω_{L1} is a linear temperature field, when the arbitrary boundary conditions of $\mathbf{u}^{L1}$ in Eq. (22) are imposed on Ω_{L1} , the temperature in Ω_{L1} can be expressed by superposing Eq. (27) for $i \in \{1, \dots, 4\}$:

$$\begin{aligned} u(\mathbf{x}) &= \mathbf{A}^{R1}(\mathbf{x}) \mathbf{u}^{L1} \\ u_{\text{explained}}^{R1} &= \mathbf{A}_{\text{explained}}^{R1} \mathbf{u}^{L1} \\ \mathbf{u}_{\text{predictor}}^{R1} &= \mathbf{A}_{\text{predictor}}^{R1} \mathbf{u}^{L1} \end{aligned} \quad (28)$$

where

(29)

$$\mathbf{A}^{R1}(\mathbf{x}) = [A_1^{R1}(\mathbf{x}) \ A_2^{R1}(\mathbf{x}) \ A_3^{R1}(\mathbf{x}) \ A_4^{R1}(\mathbf{x})] \in \mathbf{R}^{1 \times 4}$$

$$\mathbf{A}_{\text{explained}}^{R1} = [A_1^{R1}(\mathbf{x}_{\text{explained}}^{R1}) \ A_2^{R1}(\mathbf{x}_{\text{explained}}^{R1}) \ A_3^{R1}(\mathbf{x}_{\text{explained}}^{R1}) \ A_4^{R1}(\mathbf{x}_{\text{explained}}^{R1})] \in \mathbf{R}^{1 \times 4}$$

$$\mathbf{A}_{\text{predictor}}^{R1} = \begin{bmatrix} A_1^{R1}(\mathbf{x}_{\text{CP1}}^{R1}) & \cdots & A_4^{R1}(\mathbf{x}_{\text{CP1}}^{R1}) \\ \vdots & \ddots & \vdots \\ A_1^{R1}(\mathbf{x}_{\text{CP4}}^{R1}) & \cdots & A_4^{R1}(\mathbf{x}_{\text{CP4}}^{R1}) \end{bmatrix} \in \mathbf{R}^{4 \times 4}$$

The bottom figures of Fig. 3 present all entries of $\mathbf{A}_{\text{predictor}}^{R1}$. For simplicity, the entry in the i -th row and j -th column of $\mathbf{A}_{\text{predictor}}^{R1}$ is denoted A_{ij}^{R1} , i.e., $A_{ij}^{R1} = A_j^{R1}(\mathbf{x}_{\text{CPI}}^{R1})$ in Fig. 3. The first equation in (28) indicates that the temperature profile is expressed by a superposition of the four temperature distributions, $A_1^{R1}(\mathbf{x}), \dots, A_4^{R1}(\mathbf{x})$, as shown in Fig. 4 for arbitrary $\mathbf{u}^{L1}$. From Eq. (28), $\mathbf{u}^{L1}$, $u(\mathbf{x})$, and $u_{\text{explained}}^{R1}$ can be expressed as

$$\mathbf{u}^{L1} = (\mathbf{A}_{\text{predictor}}^{R1})^{-1} \mathbf{u}_{\text{predictor}}^{R1} \quad (30)$$

$$u(\mathbf{x}) = \mathbf{A}^{R1}(\mathbf{x}) (\mathbf{A}_{\text{predictor}}^{R1})^{-1} \mathbf{u}_{\text{predictor}}^{R1} = \mathbf{N}^{R1}(\mathbf{x}) \mathbf{u}_{\text{predictor}}^{R1}$$

$$u_{\text{explained}}^{R1} = \mathbf{A}_{\text{explained}}^{R1} (\mathbf{A}_{\text{predictor}}^{R1})^{-1} \mathbf{u}_{\text{predictor}}^{R1} = \mathbf{a}^{R1} \mathbf{u}_{\text{predictor}}^{R1}$$

where

$$\mathbf{N}^{R1}(\mathbf{x}) = \mathbf{A}^{R1}(\mathbf{x}) (\mathbf{A}_{\text{predictor}}^{R1})^{-1} \in \mathbf{R}^{1 \times 4} \quad (31)$$

$$\mathbf{a}^{R1} = \mathbf{A}_{\text{explained}}^{R1} (\mathbf{A}_{\text{predictor}}^{R1})^{-1} \in \mathbf{R}^{1 \times 4}$$

are the interpolating function matrix and the influence coefficient matrix for Ω_{R1} , respectively. Because the number

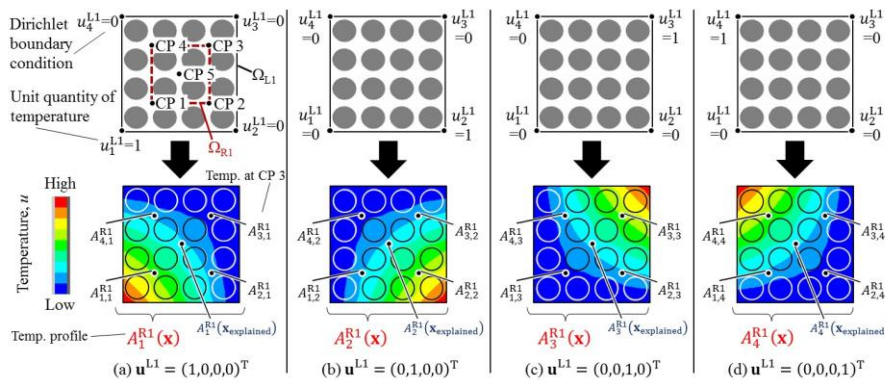


Fig. 3 Schematic illustration of obtaining all the components of matrices $\mathbf{A}^{R1}(\mathbf{x})$, $\mathbf{A}_{\text{explained}}^{R1}$, and $\mathbf{A}_{\text{predictor}}^{R1}$ in Eq. (29). Note that $A_{ij}^{R1} = A_j^{R1}(\mathbf{x}_{\text{CPI}}^{R1})$.

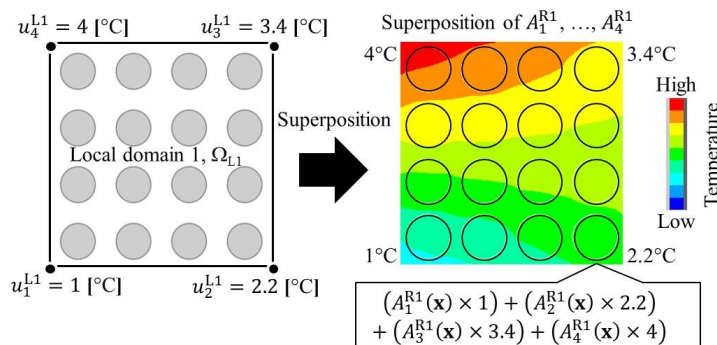


Fig. 4 Example temperature distribution in the local domain, expressed by superposition of the four temperature profiles shown in Figs. 3(a)–(d). The superposition can express the temperature profile when the local domain has an arbitrary set of temperature values

$$(\mathbf{u}^{L1} = [u_1^{L1} \ u_2^{L1} \ u_3^{L1} \ u_4^{L1}]^T).$$

of components of $\mathbf{u}_{\text{predictor}}^{\text{R1}}$ is the same as that of $\mathbf{u}^{\text{L1}}$, $(\mathbf{A}_{\text{predictor}}^{\text{R1}})^{-1}$ can be uniquely derived. Therefore $\mathbf{u}_{\text{predictor}}^{\text{R1}}$ can uniquely determine $\mathbf{u}^{\text{L1}}$ and $\mathbf{u}_{\text{explained}}^{\text{R1}}$. By dividing Ω_{L1} into sufficiently fine finite-element elements, we can obtain accurate values for $\mathbf{N}^{\text{R1}}$ and $\mathbf{a}^{\text{R1}}$. Temperature profile interpolated by precise $\mathbf{N}^{\text{R1}}$ would sufficiently well obey the governing equation of the simulated field. In addition, $\mathbf{N}^{\text{R1}}$ can reproduce complicated temperature profiles that correctly consider the microscopic heterogeneity of the structure.

The above illustration and Fig. 3 show that we must conduct the finite-element analysis four times to determine $\mathbf{N}^{\text{R1}}$ and $\mathbf{a}^{\text{R1}}$. However, they can be obtained from the result of only a single analysis of the local domain (see reference by Suzuki and Soga (2016)). It is important that an increase in the number of surrounding CPs (m_i) improves the accuracy of $u(\mathbf{x})$ and $u_{\text{explained}}^{\text{R1}}$ computed by Eq. (30).

Note that Ω_{L1} includes Ω_{R1} ($\Omega_{\text{L1}} \supset \Omega_{\text{R1}}$). Enclosing Ω_{R1} with the other unit cells is called the oversampling technique, which removes the boundary effect on Ω_{R1} (Chua, et al., 2008, Henning and Peterseim, 2013, Efendiev, et al., 2013). As shown in Fig. 1, Ω_{R1} is located inside of the entire global domain and far from the global boundary. In other words, temperature in Ω_{R1} is not strongly affected due to the existence of the global boundary. Therefore, when constructing $\mathbf{N}^{\text{R1}}$ and $\mathbf{a}^{\text{R1}}$, we need to place Ω_{R1} as far from the local domain's boundary as possible. Conversely, when we prepare $\mathbf{N}^{\text{Ri}}$ and $\mathbf{a}^{\text{Ri}}$ for the region of influence that is adjacent to the global boundary (e.g., Region 3 in Fig. 1), we need not to enclose the region with the other unit cells so that the region receives the boundary effect.

As Ω_{R1} and Ω_{R2} are the same in shape, dimension, and the number and locations of the CPs (Fig. 1), $\mathbf{N}^{\text{R1}}$ and $\mathbf{a}^{\text{R1}}$ that are derived for Ω_{R1} can be used for Ω_{R2} . Therefore,

$$u(\mathbf{x}) = \mathbf{N}^{\text{R1}}(\mathbf{x}) \mathbf{u}_{\text{predictor}}^{\text{R2}} \quad \text{for } \mathbf{x} \in \Omega_{\text{R2}} \quad (32)$$

where $\mathbf{u}_{\text{predictor}}^{\text{R2}} = [u_6 \ u_7 \ u_8 \ u_5]^T$ is the temperature value vector for the four CPs located on the boundary of Ω_{R2} . Once we obtain a single kind of interpolation function, we can compute temperature profiles in Ω_{R1} , Ω_{R2} , and other similar regions of influence. After conducting all local analyses and obtaining interpolation function matrices and influence coefficient matrices for all regions of influence, we move on to analysis of the entire global domain using the matrices, which is called the global analysis. The procedure of the global analysis is excluded here because it was briefly illustrated in Subsection 2.2.

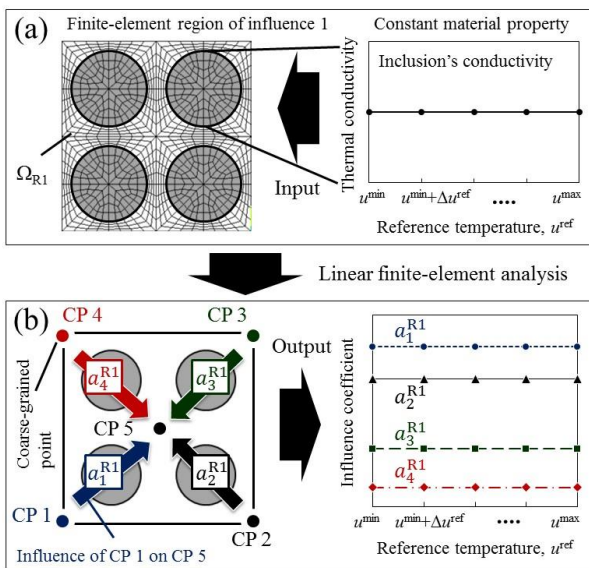


Fig. 5 (a) Example of a region of influence whose material property is constant at any temperature; and (b) the influence coefficients for the region that are not dependent on temperature.

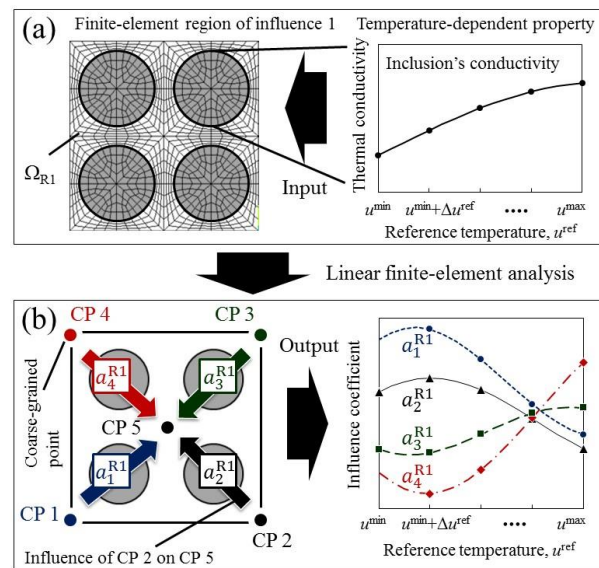


Fig. 6 (a) Region of influence whose material property changes according to temperature; and (b) the influence coefficients of the region that also depend on temperature.

2.4 SDM for nonlinear problems without iterative multiscale analysis

This subsection illustrates how to construct the interpolation functions (f^{Ri} in Eq. (11)) for two-dimensional nonlinear stationary temperature field governed by the below equation.

$$D_{11}(\mathbf{x}, u) \frac{\partial^2 u(\mathbf{x})}{\partial x_1^2} + D_{22}(\mathbf{x}, u) \frac{\partial^2 u(\mathbf{x})}{\partial x_2^2} = 0 \quad (33)$$

In the nonlinear field, the thermal conductivity has a temperature dependency.

2.4.1 Local analysis of the SDM for the nonlinear problems

As stated in Subsection 2.3 and depicted in Fig. 5, when the field is linear and its thermal conductivity is constant against the temperature, interpolation function matrix ($\mathbf{N}^{Ri}$ in Eq. (14)) and influence coefficient matrix ($\mathbf{a}^{Ri}$ in Eq. (16)) for a region of influence do not change according to the temperature. Therefore, both $\mathbf{N}^{Ri}$ and $\mathbf{a}^{Ri}$ can be determined using only one linear finite-element analysis of the i -th local domain (Ω_{Li}). We do not have to repeat the analysis when the problem is linear.

However, both $\mathbf{N}^{Ri}$ and $\mathbf{a}^{Ri}$ are dependent on the temperature when analyzing the field having nonlinear thermal conductivity, as shown in the top figure of Fig. 6. Except that the matrices have a temperature dependency, all the formulations of the nonlinear SDM analysis are the same as those of the linear SDM scheme.

$\mathbf{N}^{Ri}$ that is expressed as a function of $\mathbf{u}$ and $\mathbf{x}$ cannot be constructed using a single analysis of the local domain.

$$\mathbf{N}^{Ri}(\mathbf{x}, u) = [N_1^{Ri}(\mathbf{x}, u) \quad \cdots \quad N_{m_i}^{Ri}(\mathbf{x}, u)] \in \mathbf{R}^{1 \times m_i} \quad (34)$$

By substituting $\mathbf{x} = \mathbf{x}_{\text{explained}}^{Ri}$ (i.e., the position of the center CP in Ω_{Ri}) into the above equation, we obtain

$$\mathbf{a}^{Ri}(u) = \mathbf{N}^{Ri}(\mathbf{x}_{\text{explained}}^{Ri}, u) \in \mathbf{R}^{1 \times m_i} \quad (35)$$

Each entry of $\mathbf{a}^{Ri}(u)$ is dependent on u , as shown in the bottom right of Fig. 6. Constructing $\mathbf{N}^{Ri}(\mathbf{x}, u)$ and $\mathbf{a}^{Ri}(u)$ for arbitrary u requires an infinite amount of computational time and would be impossible.

Therefore, $\mathbf{N}^{Ri}(\mathbf{x}, u)$ and $\mathbf{a}^{Ri}(u)$ are only constructed for a finite set of discretized temperature values (referred to as the reference temperatures, u^{ref}). We must repeat the local analysis m_L times to generate $\mathbf{N}^{Ri}(\mathbf{x}, u^{\text{ref}})$ and $\mathbf{a}^{Ri}(u^{\text{ref}})$ for m_L temperature values. Now

$$u^{\text{ref}} = u^{\min}, u^{\min} + \Delta u^{\text{ref}}, u^{\min} + 2\Delta u^{\text{ref}}, \dots, u^{\max} \quad (36)$$

where

$$\Delta u^{\text{ref}} = \frac{1}{m_L - 1} (u^{\max} - u^{\min}) \quad (37)$$

We then obtain

$$\mathbf{N}^{Ri} = \mathbf{N}^{Ri}(\mathbf{x}, u^{\min}), \mathbf{N}^{Ri}(\mathbf{x}, u^{\min} + \Delta u^{\text{ref}}), \dots, \mathbf{N}^{Ri}(\mathbf{x}, u^{\max}) \quad (38)$$

and

$$\mathbf{a}^{Ri} = \mathbf{a}^{Ri}(u^{\min}), \mathbf{a}^{Ri}(u^{\min} + \Delta u^{\text{ref}}), \dots, \mathbf{a}^{Ri}(u^{\max}) \quad (39)$$

where $u^{\max}$ and $u^{\min}$ are the maximum and minimum temperatures in the entire global domain. Note that $\mathbf{N}^{Ri}(\mathbf{x}, u^{\text{ref}})$ is the interpolation matrix and $\mathbf{a}^{Ri}(u^{\text{ref}})$ is the influence coefficient matrix at u^{ref} . These matrices

are calculated using a linear finite-element analysis of the local domain that has thermal conductivities of

$$\begin{aligned} D_{11} &= D_{11}(\mathbf{x}, u^{\text{ref}}) \quad \text{for arbitrary } \mathbf{x} \in \Omega_{Li} \\ D_{22} &= D_{22}(\mathbf{x}, u^{\text{ref}}) \end{aligned} \quad (40)$$

u^{ref} is constant throughout Ω_{Li} . Therefore, we determine the material properties used in the analysis by assuming that the entire local domain has a uniform temperature distribution:

$$u(\mathbf{x}) = u^{\text{ref}} \quad \text{for arbitrary } \mathbf{x} \in \Omega_{Li} \quad (41)$$

In general, the local domain has a non-uniform temperature and this assumption, shown in Equations (40) and (41), causes errors in $\mathbf{N}^{Ri}(\mathbf{x}, u^{\text{ref}})$ and $\mathbf{a}^{Ri}(u^{\text{ref}})$. Although the error can be reduced by increasing the number of discretized temperature values, m_L in Eq. (37), it is non-negligible when the material nonlinearity is notably strong and the local domain has a steep temperature gradient. To investigate the errors, we solved various numerical examples using the proposed methods (see Section 3).

The assumption of Equations (40) and (41) indicates that the analysis of Ω_{Li} (i.e., local analysis of the SDM) can be conducted employing the linear FEM solver even if the analytical field has a nonlinear thermal property. Each local domain has different thermal conductivities ($D_{11}(\mathbf{x}, u^{\text{ref}})$ and $D_{22}(\mathbf{x}, u^{\text{ref}})$), calculated using different reference temperatures (u^{ref}) and Eq. (40). Consequently, $\mathbf{N}^{Ri}(\mathbf{x}, u^{\text{ref}})$ and $\mathbf{a}^{Ri}(u^{\text{ref}})$ can be obtained in a similar way to $\mathbf{N}^{R1}(\mathbf{x})$ and $\mathbf{a}^{R1}$ in Eq. (31) that are illustrated in Subsection 2.3.

Because we obtain the temperature distribution throughout the global domain by connecting the local distributions to each other, there are discontinuities where regions of influence meet. This method greatly decreases the computational costs, but leads to non-negligible errors under certain conditions.

The summary of the above procedure is given below and in the left figure of Fig. 7.

1. Set the reference temperature

$$u^{\text{ref}} = u^{\min} \quad (42)$$

and substitute it into Eq. (40) to calculate the temperature-dependent material property.

2. Substitute $u(\mathbf{x}) = u^{\text{ref}}$ for $\mathbf{x} \in \Omega_{Li}$ into Eq. (33). Note that Eq. (33) becomes a linear differential equation, because u^{ref} is assumed to be constant throughout Ω_{Li} .
3. Numerically solve Eq. (33) by applying the linear FEM.
4. Construct the interpolating function matrix, $\mathbf{N}^{Ri}(\mathbf{x}, u^{\text{ref}})$, from the FEM result.
5. Construct the influence coefficient matrix, $\mathbf{a}^{Ri}(u^{\text{ref}})$, from $\mathbf{N}^{Ri}(\mathbf{x}, u^{\text{ref}})$ and Eq. (35).
6. Reset

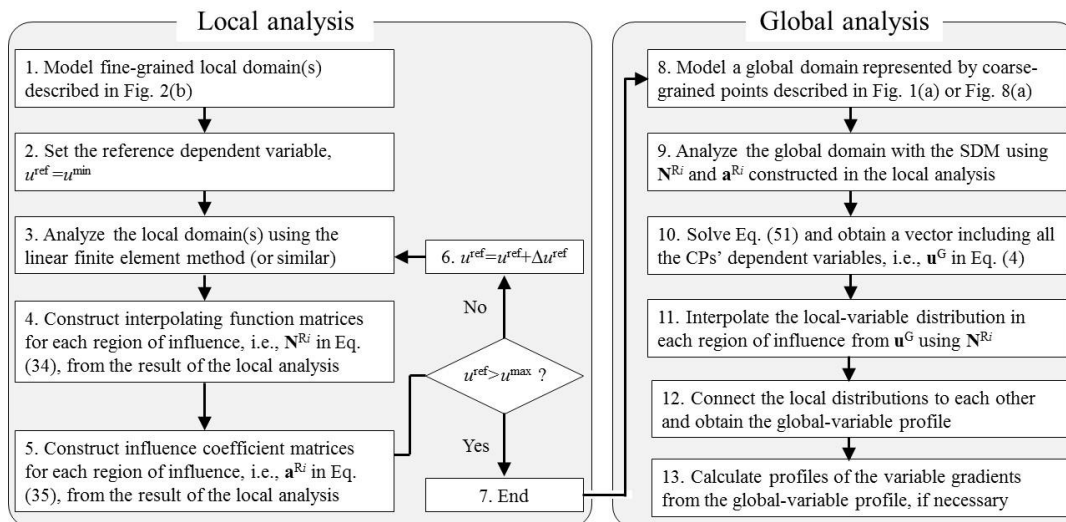


Fig. 7 Proposed seamless-domain method (SDM) for nonlinear elliptic boundary value problems.

$$\mathbf{u}^{\text{ref}} = \mathbf{u}^{\text{ref}} + \Delta \mathbf{u}^{\text{ref}} \quad (43)$$

and redo Steps 3–6 until $\mathbf{u}^{\text{ref}}$ increases to $\mathbf{u}^{\text{max}}$.

The above procedure gives m_L values of $\mathbf{N}^{\text{Ri}}$ and $\mathbf{a}^{\text{Ri}}$ for m_L reference temperature values, using the results of m_L linear local analyses. In other words, the local analysis of the SDM analyzes the local domain using the linear FEM m_L times, to prepare $\mathbf{N}^{\text{Ri}}$ and $\mathbf{a}^{\text{Ri}}$ for the subsequent nonlinear global analysis. Therefore, the cost of the calculations in the local analysis is linearly proportional to m_L .

2.4.2 Global analysis of the SDM for the nonlinear problems

We conduct the global analysis using the interpolating function matrix ($\mathbf{N}^{\text{Ri}}$ in Eq. (34)), and the influence coefficient matrix ($\mathbf{a}^{\text{Ri}}$ in Eq. (35)). These are both constructed during the local analysis.

To illustrate how to implement the global analysis, we consider a small domain depicted in Fig. 8(a) as an example of global domain, Ω_G . Ω_G is represented by only 25 CPs (i.e., $n = 25$). Its boundary is Γ_G . Each CP has temperature as a dependent variable. The number of degrees of freedom (DOFs) in Ω_G is the same as that of the CPs, i.e., n . Ω_G has no element, grid, mesh, or cell, and is represented by a small number of CPs that are depicted as black circles in Fig. 8(a). This means that Ω_G does not require an individual cylindrical inclusion and base material. Because the CPs are only placed at the corners of each unit cell, there are considerably fewer DOFs than in a direct (exact) finite-element global domain (which constructs the individual inclusion and base material models). $\mathbf{u}^G$ is a vector comprising temperatures at all the CPs in Ω_G , and has 25 components. That is,

$$\mathbf{u}^G = [u_1 \quad \cdots \quad u_{25}]^T \quad (44)$$

where u_i is temperature value at CP i .

We must first determine all the components of $\mathbf{u}^G$, as in Steps 9 and 10 of the flowchart in Fig. 7. Ω_G consists of many regions of influence (Ω_{Ri} for $i \in \{1, \dots, 25\}$) centered on each of the CPs. In the SDM, the temperature at the center CP of each Ω_{Ri} is expressed as a linear combination of the influence coefficients and surrounding CP temperatures. For instance, the temperature at CP 10 in the first region of influence (the bold frame in Fig. 8(b)) is calculated from the temperatures at CPs 6, 7, 13, 14 (i.e., $\mathbf{u}_{\text{predictor}}^{\text{R1}} = [u_{13} \quad u_{14} \quad u_7 \quad u_6]^T$). That is,

$$\begin{aligned} u_{10} &= \mathbf{a}^{\text{R1}} \mathbf{u}_{\text{predictor}}^{\text{R1}} \\ &= [a_1^{\text{R1}}(u_{13}^{\text{ref}}) \quad a_2^{\text{R1}}(u_{14}^{\text{ref}}) \quad a_3^{\text{R1}}(u_7^{\text{ref}}) \quad a_4^{\text{R1}}(u_6^{\text{ref}})] [u_{13} \quad u_{14} \quad u_7 \quad u_6]^T \\ &= a_1^{\text{R1}}(u_{13}^{\text{ref}}) u_{13} + a_2^{\text{R1}}(u_{14}^{\text{ref}}) u_{14} + a_3^{\text{R1}}(u_7^{\text{ref}}) u_7 + a_4^{\text{R1}}(u_6^{\text{ref}}) u_6 \end{aligned} \quad (45)$$

where $a_1^{\text{R1}}(u_{13}^{\text{ref}})$ is the influence coefficient of u_{13} . u_i^{ref} is the reference temperature for CP i . There are several options for u_i^{ref} . In this article, we considered the following two cases for all the numerical examples (see Section 3).

• u_i^{ref} is the temperature at CP i , that is,

$$u_i^{\text{ref}} = u_i \quad (46)$$

In this case,

$$u_{13}^{\text{ref}} = u_{13} \quad (47)$$

• u_i^{ref} is the average value of the temperatures at m_i surrounding CPs in Region i (Ω_{Ri}), that is,

$$u_i^{\text{ref}} = \frac{1}{m_i} \sum_j u_j \quad (48)$$

In this case,

$$u_{13}^{\text{ref}} = \frac{1}{4}(u_6 + u_7 + u_{13} + u_{14}) \quad (49)$$

Similarly, focusing on Region 2 (the dashed frame in Fig. 8(b)), u_{13}^G is expressed as

$$\begin{aligned} u_{13} &= \mathbf{a}^{\text{R2}} \mathbf{u}_{\text{predictor}}^{\text{R2}} \\ &= a_1^{\text{R2}}(u_{16}^{\text{ref}})u_{16} + a_2^{\text{R2}}(u_{17}^{\text{ref}})u_{17} + a_3^{\text{R2}}(u_{10}^{\text{ref}})u_{10} + a_4^{\text{R2}}(u_9^{\text{ref}})u_9 \end{aligned} \quad (50)$$

It is important to note that the influence coefficient matrix for Region 2 is the same as that for Region 1, i.e., $\mathbf{a}^{\text{R1}} = \mathbf{a}^{\text{R2}}$. Because the global domain has a periodic microstructure (see Fig. 8(a)), the same influence coefficients can be used for many regions of influence. However, note that each component of $\mathbf{a}^{\text{R1}}$ in Eq. (45) is not necessarily equal to that in Eq. (50), because the components of $\mathbf{a}^{\text{R1}}$ change according to the reference temperature, u_i^{ref} . By formulating equations similar to Equations (45) and (50) for all 25 regions of influence, we obtain simultaneous nonlinear equations:

$$\mathbf{a}^G \mathbf{u}^G = \mathbf{u}^G \quad (51)$$

where

$$\mathbf{a}^G = \begin{bmatrix} 0 & a_{1,2}^G & a_{1,3}^G & \cdots & 0 & 0 & 0 \\ a_{2,1}^G & 0 & a_{2,3}^G & a_{2,4}^G & \ddots & 0 & 0 \\ a_{3,1}^G & a_{3,2}^G & 0 & a_{3,4}^G & \cdots & \ddots & 0 \\ \vdots & a_{4,2}^G & a_{4,3}^G & 0 & \cdots & \cdots & \vdots \\ 0 & \ddots & a_{5,3}^G & a_{5,4}^G & \ddots & \cdots & a_{23,25}^G \\ 0 & 0 & \ddots & \cdots & \cdots & \ddots & a_{24,25}^G \\ 0 & 0 & 0 & \cdots & a_{25,23}^G & a_{25,24}^G & 0 \end{bmatrix} \in \mathbf{R}^{25 \times 25} \quad (52)$$

$\mathbf{a}^G$ is a band matrix established by assembling all the influence coefficients in the correct order. $a_{i,j}^G$ is the entry in the i -th row and j -th column of $\mathbf{a}^G$, and denotes the degree of the influence of u_j^G (temperature at CP j) on u_i^G (temperature at CP i). Note that Eq. (52) indicates that $a_{i,i}^G = 0$, because u_i^G does not affect itself. Additionally, $a_{i,j}^G$ is a function of u_j^{ref} . The tenth and thirteenth rows of Eq. (51) are

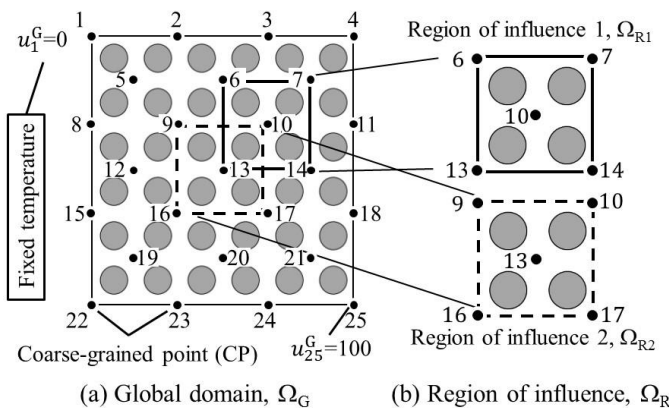


Fig. 8 (a) Global domain represented by 25 coarse-grained points; and (b) two regions of influence extracted from the global domain.

$$\begin{array}{c}
 1_{\text{st}} \quad \cdots \quad 5_{\text{th}} \quad 6_{\text{th}} \quad 7_{\text{th}} \quad 8_{\text{th}} \quad 9_{\text{th}} \quad 10_{\text{th}} \quad 11_{\text{th}} \quad 12_{\text{th}} \quad 13_{\text{th}} \quad 14_{\text{th}} \quad 15_{\text{th}} \quad 16_{\text{th}} \quad 17_{\text{th}} \quad \cdots \quad 25_{\text{th}} \\
 \begin{array}{c}
 1_{\text{st}} \\
 \vdots \\
 10_{\text{th}} \\
 \vdots \\
 13_{\text{th}} \\
 \vdots \\
 25_{\text{th}}
 \end{array}
 \begin{bmatrix}
 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 0 & \cdots & 0 & a_4^{R1}(u_6^{\text{ref}}) & a_3^{R1}(u_7^{\text{ref}}) & 0 & 0 & 0 & 0 & a_1^{R1}(u_{12}^{\text{ref}}) & a_2^{R1}(u_{13}^{\text{ref}}) & 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & \cdots \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 0 & \cdots & 0 & 0 & 0 & 0 & a_4^{R1}(u_9^{\text{ref}}) & a_3^{R1}(u_{10}^{\text{ref}}) & 0 & 0 & 0 & 0 & 0 & a_1^{R1}(u_{16}^{\text{ref}}) & a_2^{R1}(u_{17}^{\text{ref}}) & \cdots & \cdots & \cdots & \cdots \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
 \end{bmatrix}
 \begin{bmatrix}
 u_1 \\
 \vdots \\
 u_{10} \\
 \vdots \\
 u_{13} \\
 \vdots \\
 u_{25}
 \end{bmatrix}
 =
 \begin{bmatrix}
 u_1 \\
 \vdots \\
 u_{10} \\
 \vdots \\
 u_{13} \\
 \vdots \\
 u_{25}
 \end{bmatrix}
 \end{array}
 \quad (53)$$

Note that $\mathbf{a}^G$ appears on both sides of Eq. (51), so the equation cannot be solved. We must therefore impose boundary conditions on Ω_G , so we can solve for $\mathbf{u}^G$. For instance, consider the temperatures at CPs 1 and 25, i.e., $u_1 = 0, u_{25} = 100$. Then, we can solve Eq. (53) and determine the temperatures for CPs 1–25. Although $\mathbf{u}^G$ is constant, $\mathbf{a}^G$ is dependent on $u_1^{\text{ref}}, \dots, u_{25}^{\text{ref}}$. Additionally, $u_1^{\text{ref}}, \dots, u_{25}^{\text{ref}}$ are calculated from $\mathbf{u}^G$ using Equations (46) or (48). Consequently, $\mathbf{a}^G$ depends on $\mathbf{u}^G$, Eq. (53) is nonlinear, and we need repeated calculations to obtain the solution of $\mathbf{u}^G$.

After determining $\mathbf{u}^G$, the detailed temperature distribution in each region of influence is interpolated from $\mathbf{u}^G$ using $\mathbf{N}^{\text{Ri}}$ in Eq. (34) (Step 11 of Fig. 7). For instance, the temperature distribution in Region 1 (bold frame in Fig. 8(b)) is interpolated from the temperatures at CPs 6, 7, 13, 14. That is,

$$\begin{aligned}
 u(\mathbf{x}) &= \mathbf{N}^{\text{R1}} \mathbf{u}_{\text{predictor}}^{\text{R1}} & \text{for } \mathbf{x} \in \Omega_{\text{R1}} & \quad (54) \\
 &= \left[N_1^{\text{R1}}(\mathbf{x}, u_{13}^{\text{ref}}) \quad N_2^{\text{R1}}(\mathbf{x}, u_{14}^{\text{ref}}) \quad N_3^{\text{R1}}(\mathbf{x}, u_7^{\text{ref}}) \quad N_4^{\text{R1}}(\mathbf{x}, u_6^{\text{ref}}) \right] \begin{bmatrix} u_{13} & u_{14} & u_7 & u_6 \end{bmatrix}^T
 \end{aligned}$$

Similarly, the temperature profile in Region 2 (dashed frame in Fig. 8(b)) is expressed as

$$\begin{aligned}
 u(\mathbf{x}) &= \mathbf{N}^{\text{R2}} \mathbf{u}_{\text{predictor}}^{\text{R2}} & \text{for } \mathbf{x} \in \Omega_{\text{R2}} & \quad (55) \\
 &= \left[N_1^{\text{R2}}(\mathbf{x}, u_{16}^{\text{ref}}) \quad N_2^{\text{R2}}(\mathbf{x}, u_{17}^{\text{ref}}) \quad N_3^{\text{R2}}(\mathbf{x}, u_{10}^{\text{ref}}) \quad N_4^{\text{R2}}(\mathbf{x}, u_9^{\text{ref}}) \right] \begin{bmatrix} u_{16} & u_{17} & u_{10} & u_9 \end{bmatrix}^T
 \end{aligned}$$

Note that the interpolation function matrices for Regions 1 and 2 are the same each other, i.e., $\mathbf{N}^{\text{R1}} = \mathbf{N}^{\text{R2}}$, except for the reference temperature u_i^{ref} . After interpolating the local-temperature distributions ($u(\mathbf{x})$) in all regions, we can

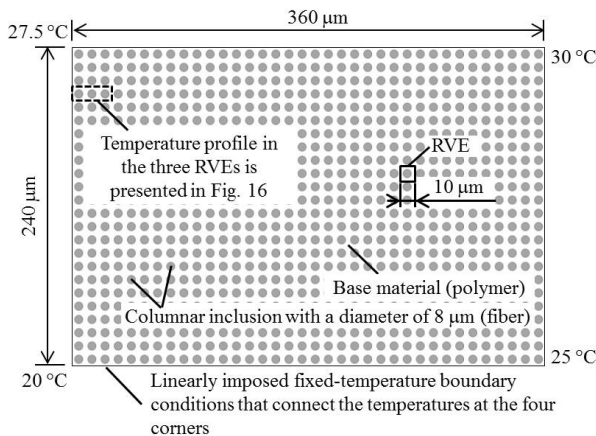


Fig. 9 Heterogeneous global domain for the example problems, which has many columnar inclusions. A linear temperature profile that linearly connects the temperatures at the four corners is imposed on the domain's outer frame as fixed-temperature boundary conditions.

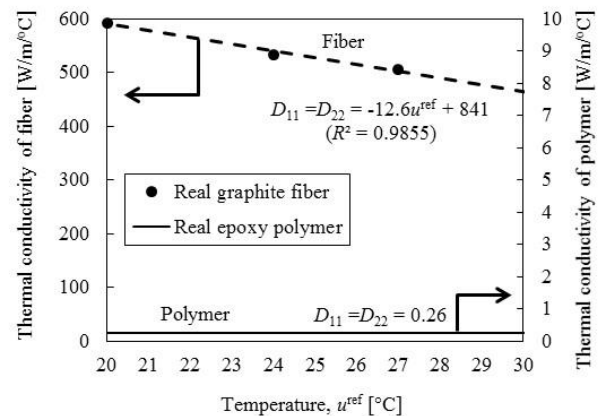


Fig. 10 Thermal conductivity of the real heterogeneous material, graphite fiber (which has a weak temperature dependency) and the conductivity of the actual epoxy polymer (which is assumed independent of the temperature).

combine them to obtain the temperature distribution throughout the global domain (Step 12 of Fig. 7). Then, we can calculate the global profiles of the temperature gradient using the global-temperature profile, if necessary (Step 13).

3. Numerical examples

To verify the accuracy of the calculations and the computational costs, we analyzed example stationary heat conduction problems. We compared two methods:

- the nonlinear SDM with quadrangular regions of influence, and
- the ordinary nonlinear FEM with a sufficient number of four-node elements (using ANSYS Version 15.0).

The FEM will produce accurate solutions because the mesh is sufficiently fine. We tested the practical efficiency of the SDM by comparing its results with the FEM solutions. In this study, we implemented the local analyses of the SDM using a conventional linear FEM (ANSYS Version 15.0), but other techniques may be used.

3.1 Problem statement

3.1.1 Real heterogeneous material

The target structure is a graphite fiber/epoxy polymer composite, which is a heterogeneous material and is described in Fig. 9. The structure has a huge number of fibers in the epoxy polymer matrix, which are approximately 5–15 μm in diameter. Fiber composites have been used as structural materials in commercial automobiles and aircraft, because they are light and very strong and stiff.

A heterogeneous solid is assumed to have a periodic microstructure (unit cell) with a graphite fiber, as depicted in Fig. 9. Perfect bonding assumes to exist between the fibers and the base polymer.

Figure 10 shows the thermal conductivities of the fiber and polymer. The polymer has a constant thermal conductivity that is much lower than the fiber. There have been few studies regarding the temperature-dependent property of the graphite fiber, so we only obtained the thermal conductivities from 20 to 30 $^{\circ}\text{C}$. We linearly interpolated the experimental conductivity values in (Zhang, et al., 2000), as shown in Fig. 10. The coefficient of determination, R^2 , is a measure of how well the trend line fits the data. In this case, $R^2=0.986$, which is sufficiently high. The conductivity of the fiber at 30 $^{\circ}\text{C}$ is 130 $\text{W/m}^{\circ}\text{C}$ (28.1 %) lower than that at 20 $^{\circ}\text{C}$. This means that the fiber does not have a strong temperature dependency. We fixed the temperatures in the bottom-left, bottom-right, top-right, and top-left corners to 20, 25, 30, and 27.5 $^{\circ}\text{C}$, respectively. Therefore, the minimum and maximum temperatures in the global domain were $u^{\min}=20, u^{\max}=30$. The outer frame of the global domain had a linear temperature profile connecting the temperatures at the corners (Fig. 9).

In addition to the SDM analysis, this example problem was solved by the homogenization method (Terada, et al., 2000, Kaczmarczyk, et al., 2010). In the homogenization process, the RVE shown in Fig. 9 was analyzed employing a nonlinear FEM solver (ANSYS software).

First, the RVE was meshed into two-dimensional four-node thermal solid elements (PLANE55 in ANSYS). There were 929 nodes and 896 elements in total. So-called periodic boundary conditions were imposed on the RVE to obtain the homogenized thermal conductivity of the RVE, D_{11} and D_{22} in Eq. 33 ($D_{11}=D_{22}$ in this case). We conducted the FEM analysis ten times and obtained the homogenized values of D_{11} at 20.5, 21.5, $\dots$, 29.5 $^{\circ}\text{C}$. The temperature dependency of D_{11} was expressed as a linear approximation. The adjusted R-square was 0.993 and the relation between D_{11} and temperature can be regarded as linear.

Second, we analyzed a homogenized entire structure having the homogenized conductivity using the nonlinear FEM solver. The homogenized model is represented by the same number of nodes as the CPs of the SDM global model (i.e., 925 points). Additionally, the model had 24-by-36 elements (PLANE55 in ANSYS) and the number of elements was equivalent to that of the RVEs. Therefore, each element corresponded to a single RVE and behaved as a standard four-node element.

3.1.2 Imaginary materials with strong material nonlinearities

To investigate the relationship between the analytical accuracy and the strength of the temperature dependencies in the material (i.e., the degree of the nonlinearity in the numerical problem), we applied our method to two kinds of imaginary materials. Figures 11 and 12 present the thermal conductivities of these two imaginary materials. The temperature dependencies of the conductivities are much larger than the actual fiber composite, which leads to stronger material nonlinearities. In particular, the second imaginary material had the largest material nonlinearity because the

inclusion's conductivity at 20 °C was ten times larger than that at 30 °C. The boundary conditions, shape, and dimensions of the global domain were the same as those of the real material, as presented Subsection 3.1.1.

3.2 Procedure of the SDM for the nonlinear problem

3.2.1 Local analysis

Details of the procedure for the local analysis were presented in Subsection 2.4.1. We analyzed a local domain consisting of four-by-four unit cells extracted from the global domain (Fig. 13(a)), to construct the regions of influence. For all the example problems in this section, we first prepared nine kinds of regions of influence, as depicted in Fig. 14. However, after generating three regions (internal (Ω_{R1}), middle-left (Ω_{R2}), and bottom-left (Ω_{R3})), we can easily obtain the other six regions (middle-right, top-middle, bottom-middle, top-left, top-right, and bottom-right) by considering the geometric symmetries. As depicted in Fig. 13(b), the internal two-by-two unit cells in the dashed frame were used as the internal region of influence; the area enclosed by the bold frame forms a neighboring region. Enclosing regions with the other unit cells is called the oversampling technique, which removes the boundary effect on the regions (Chua, et al., 2008, Henning and Peterseim, 2013, Efendiev, et al., 2013). The CPs that surround the internal region (Fig. 13(b)), the middle-left region (Fig. 13(c)), and the bottom-left region (Fig. 13(d)) are $m_1 = 8$, $m_2 = 9$, and $m_3 = 8$, respectively. We divided the local domains (Ω_{Li} for $i \in \{1, \dots, 3\}$) into a sufficient number of finite-element meshes (linear quadrangular elements). We then analyzed it using the fixed-temperature boundary conditions of $\mathbf{u}^{Li}$ on the m_i CPs on the boundary of Ω_{Li} . This determined the interpolating function matrix ($\mathbf{N}^{Ri}(\mathbf{x}, u^{\text{ref}})$), and the influence coefficient matrix ($\mathbf{a}^{Ri}(u^{\text{ref}})$). Ω_{Li} had 14,465 nodes. The outer frame of Ω_{Li} had a linear temperature profile that connected the m_i temperatures denoted as $\mathbf{u}^{Li}$.

Note that $\mathbf{N}^{Ri}(\mathbf{x}, u^{\text{ref}})$ and $\mathbf{a}^{Ri}(u^{\text{ref}})$ depend on the reference temperature (u^{ref}), because the thermal conductivities of the target materials change according to u^{ref} . Therefore, we prepared m_L values of $\mathbf{N}^{Ri}(\mathbf{x}, u^{\text{ref}})$ and $\mathbf{a}^{Ri}(u^{\text{ref}})$ for m_L reference temperature values. In this study, we tested two cases: $m_L = 10$ or 20. The minimum and maximum temperatures for the problems were 20 and 30 °C. For $m_L = 10$, we repeated the linear FEM analysis of Ω_{Li} ten times to construct $\mathbf{N}^{Ri}(\mathbf{x}, u^{\text{ref}})$ and $\mathbf{a}^{Ri}(u^{\text{ref}})$ at $u^{\text{ref}} = 20.5, 21.5, 22.5, \dots, 29.5$. For $m_L = 20$, $u^{\text{ref}} = 20.25, 20.75, 21.25, 21.75, \dots, 29.75$, which required 20 linear FEM analyses.

3.2.2 Global analysis

The global domain of the SDM (Ω_G) was represented by 925 CPs (i.e., $n = 925$), for each example. We first constructed nonlinear algebraic equations such as Eq. (51), to determine the temperatures at all the CPs ($\mathbf{u}^G$). The $a_{i,j}^G$ component of $\mathbf{a}^G$ is a function of u_i^{ref} , and we tested two kinds of u_i^{ref} . After solving the algebraic equations, we interpolated the temperature distribution in the global domain from $\mathbf{u}^G$ using $\mathbf{N}^{Ri}(\mathbf{x}, u^{\text{ref}})$.

3.3 Results

3.3.1 Computational time

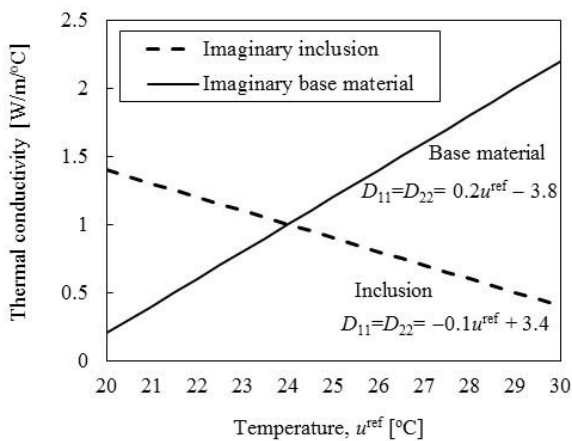


Fig. 11 Thermal conductivities of Imaginary Heterogeneous Material 1.

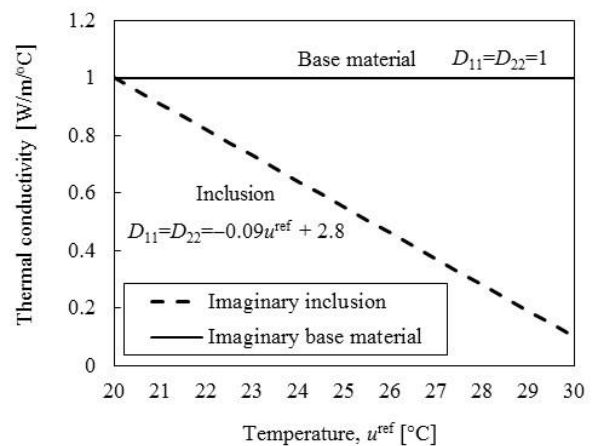


Fig. 12 Thermal conductivities of Imaginary Heterogeneous Material 2. The inclusion's conductivity has a considerably strong temperature dependency.

The total times taken to solve the problems are listed in Table 1. Because the shapes and dimensions of the target structures are the same in all the problems, they have comparable computational times. For the FEM, the results include the times taken to mesh the domains and to solve the nonlinear equations using iterative calculations. For the SDM, we present the total time taken for the local and global analyses. The ordinary FEM required 82 times the computational time of the SDM to solve the same problem when $m_L = 10$, and 44 times when $m_L = 20$. Here, m_L is the number of iterations of the local analysis of the SDM.

3.3.2 Real heterogeneous material

Figure 15 presents distributions of the temperature (u) and temperature gradients ($\partial u/\partial x_1$ and $\partial u/\partial x_2$) for the actual heterogeneous material (graphite fiber/epoxy polymer composite). In the direct FEM (Fig. 15(i)), the fiber and polymer were separately modeled and divided into fine meshes. There were 775,105 nodes and 774,144 elements. Therefore, we consider that the contours of Fig. 15(i) are exactly correct. The SDM model had only 925 CPs (Fig. 15(ii)). The homogenized finite-element model also had 925 nodes (Fig. 15(iii)). If the SDM result (or the homogenized FEM result) agreed with the direct FEM result, we considered that the calculation error was zero.

Note that the contours in Fig. 15(iii) are the results obtained from the homogenized finite-element model, not the localized results. Needless to say, each homogenized finite element, which corresponded to the RVE and behaved as a standard four-node element (Subsection 3.1.1), had a simple temperature distribution and did not take into consideration the difference in thermal conductivity between the fiber and polymer.

Figure 16(i)(ii) give partial magnified images of Fig. 15(i)(ii) showing detailed distributions of the temperature and its gradients in the three RVEs that are enclosed by the black dashed frame in Fig. 9. Figure 16(iii) presents local profiles calculated from the results of the finite-element analysis of the homogenized composite. Temperature of the black dashed frame, which is shown in Fig. 9, in the homogenized finite-element model was respectively imposed on the outer frame of each of the three heterogeneous RVEs as a boundary condition. In the heterogeneous RVEs, the fibers and polymer were respectively divided into fine finite-element meshes. This process was conducted in a standard procedure of the zooming technique. The local temperature distribution (the top figure of Fig. 16(iii)) was exactly continuous at the interfaces between the RVEs and sufficiently agree with that of the direct FEM (Fig. 16(i)). However, the temperature gradients were different from those of the direct FEM. Especially, the gradient in the x_1 direction was greatly discontinuous at the interfaces of the RVEs.

Conversely, the SDM model (925 CPs, $m_L = 20$, and reference temperatures calculated using Eq. (48)) generated sufficiently precise and sufficiently continuous profiles of temperature gradients (the bottom two figures of Fig. 16(ii)) that cannot be distinguished from those of the direct FEM model with 775,105 nodes (Fig. 16(i)) by their appearances.

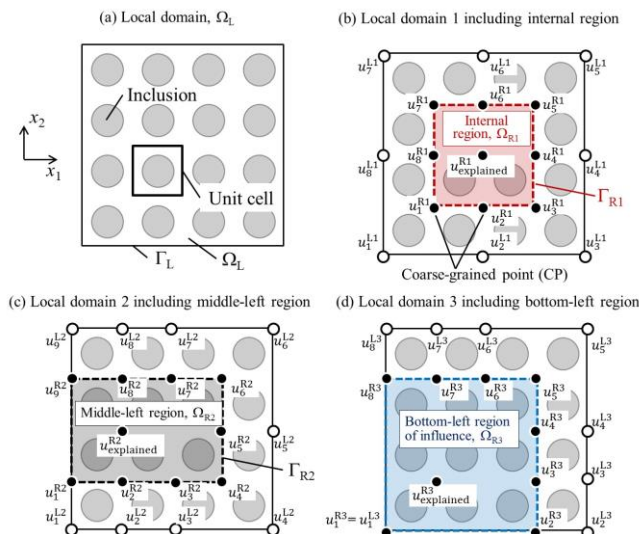


Fig. 13 Layout of the coarse-grained points (CPs) in the local domains that are composed of four-by-four unit cells, for the example problems: (a) local domain; (b) local domain including the internal region of influence; (c) local domain including the middle-left region; and (d) local domain including the bottom-left region.

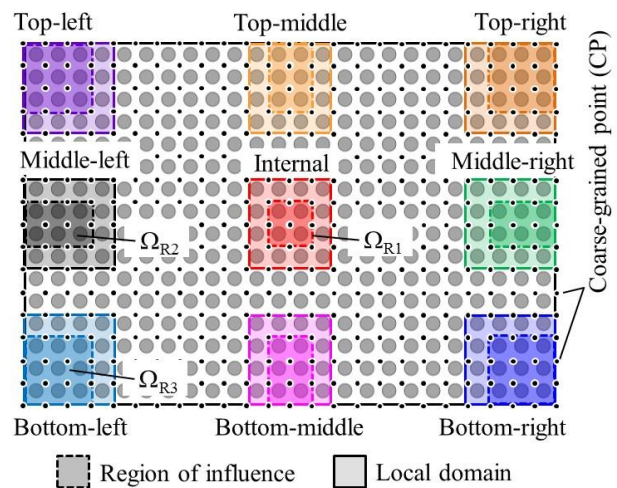


Fig. 14 Nine kinds of regions of influence for the global analysis of the SDM. The bold frame is the local domain, and the dashed frame is the region of influence.

However, as shown in the right RVE of Fig. 16(ii), temperature of the SDM was partially and slightly different from the direct FEM. Consequently, we believe that the proposed SDM can sufficiently accurately solve the nonlinear problem when analyzing the actual fiber composite material.

Table 2 presents the temperature difference compared to the result of the direct FEM. The SDMs with four different

Table 1 Comparison of total computational times.

Item	DOF in Ω_G	Number of $\mathbf{a}^{Rl}(u^{ref}), m_l$	Total calculation time
FEM	775,105		2031 sec
SDM	925	10	24.7 sec
SDM	925	20	46.4 sec

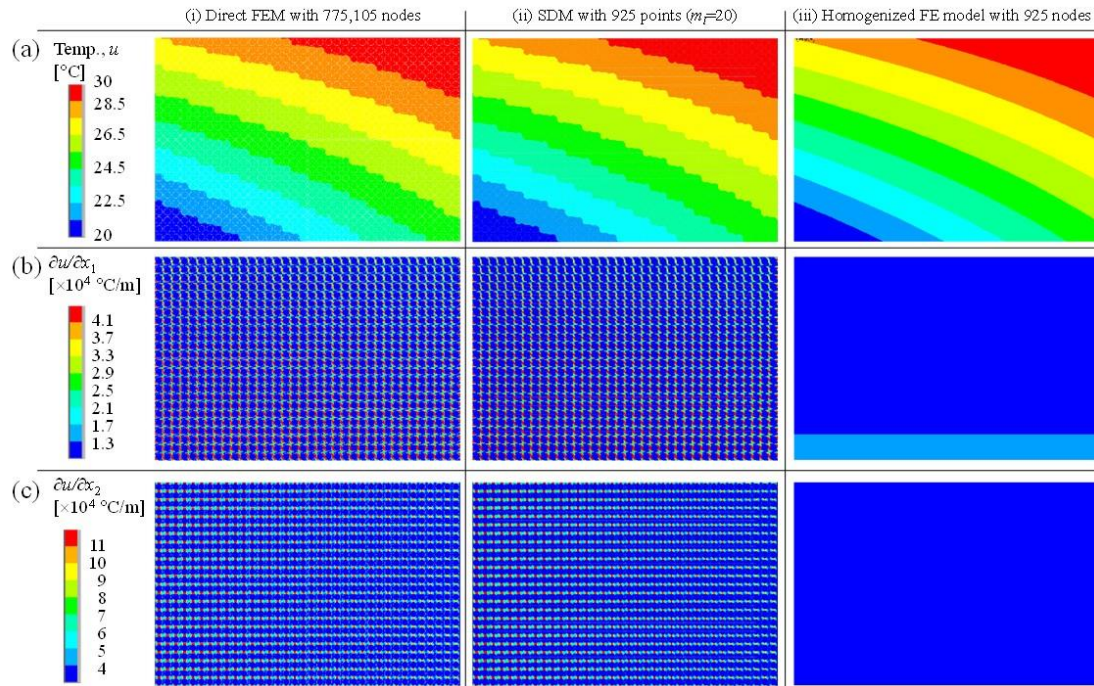


Fig. 15 Calculated contours for the actual heterogeneous material (graphite/epoxy composite in Fig. 10): (a) temperature, u ; and the temperature gradients (b) $\partial u/\partial x_1$ and (c) $\partial u/\partial x_2$. The contours presented in (iii) are obtained from the homogenized structure, not the localized profiles.

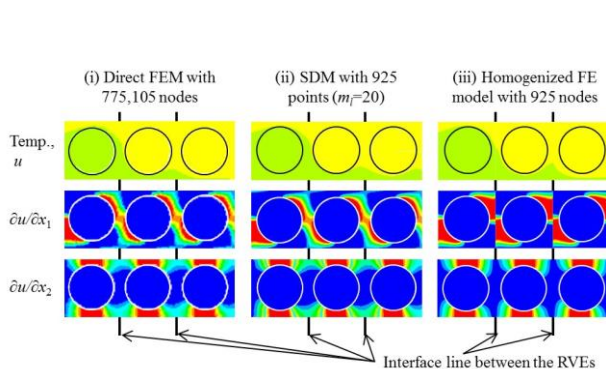


Fig. 16 Partial magnified images of Fig. 15 showing detailed distributions of the temperature and its gradients in the three RVEs that are enclosed by the black dashed frame in Fig. 9. The local distributions calculated from the results of the finite-element analysis of the homogenized composite are presented in (iii).

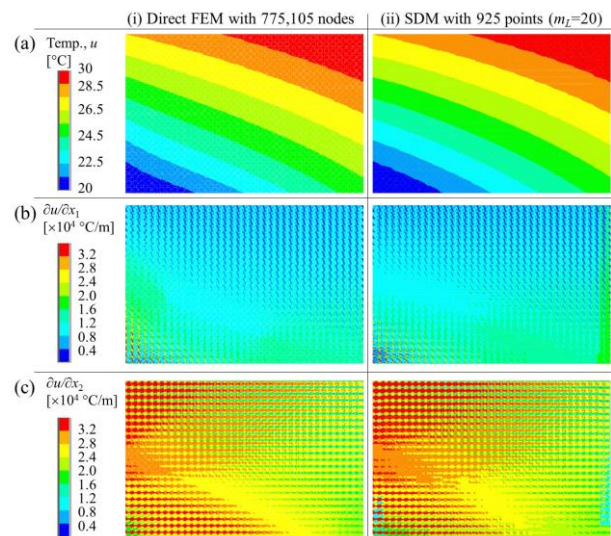


Fig. 17 Calculated contours for Imaginary Heterogeneous Material 1 (shown in Fig. 11): (a) temperature, u ; and temperature gradients (b) $\partial u/\partial x_1$ and (c) $\partial u/\partial x_2$.

conditions gave a solution that was as accurate as the FEM. In other words, m_L and the method used to derive the reference temperatures did not have significant influences on the accuracy for this problem. The average temperature difference over all CPs (i.e., 925 points) was approximately $1.18 \times 10^{-3} \text{ }^\circ\text{C}$, which is sufficiently small when we consider that a temperature change of $10 \text{ }^\circ\text{C}$ (from 20 to $30 \text{ }^\circ\text{C}$) was generated over the global domain. The average temperature difference over the 925 nodes of the homogenized model was $8.32 \times 10^{-5} \text{ }^\circ\text{C}$ and smaller by nearly one digit than the SDM.

Consequently, the homogenized model generated better global temperature (i.e., temperature values at the 925 points) than the SDM when analyzing the entire structure; the SDM gave better localized distributions of temperature and its gradients than the homogenization scheme when analyzing the partial regions extracted from the entire structure.

3.3.3 Imaginary materials with strong material nonlinearities

Table 3 contains the differences between the temperatures calculated by the two methods, when analyzing the first

Table 2 Comparison of analytical accuracies compared with the direct FEM solution when analyzing the real heterogeneous material (Fig. 10).

Item	Number of $\mathbf{a}^{Ri}(\mathbf{u}^{\text{ref}})$	Reference temp. for CP i	Average difference in temperature compared with direct FEM	Maximum difference in temperature compared with direct FEM
Symbol	m_L	u_i^{ref}	u	
Unit			$^\circ\text{C}$	
SDM	10	u_i^G (Eq. (46))	1.6854×10^{-3}	4.5394×10^{-3}
SDM	10	Average (Eq. (48))	1.6881×10^{-3}	4.5459×10^{-3}
SDM	20	u_i^G (Eq. (46))	1.1840×10^{-3}	3.468×10^{-3}
SDM	20	Average (Eq. (48))	1.1839×10^{-3}	3.467×10^{-3}
Homogenization			8.323×10^{-5}	1.00×10^{-3}

Table 3 Comparison of analytical accuracies between the SDM solutions and the direct FEM solution when analyzing Imaginary Material 1 (Fig. 11).

Item	Number of $\mathbf{a}^{Ri}(\mathbf{u}^{\text{ref}})$	Reference temp. for CP i	Average difference in temperature between SDM and direct FEM	Maximum difference in temperature between SDM and direct FEM
Symbol	m_L	u_i^{ref}	u	
Unit			$^\circ\text{C}$	
SDM	10	u_i^G (Eq. (46))	0.08334	0.20181
SDM	10	Average (Eq. (48))	0.08334	0.20181
SDM	20	u_i^G (Eq. (46))	0.08370	0.20210
SDM	20	Average (Eq. (48))	0.08370	0.20210

Table 4 Comparison of analytical accuracies between the SDM solutions and the direct FEM solution when analyzing Imaginary Material 2 (Fig. 12).

Item	Number of $\mathbf{a}^{Ri}(\mathbf{u}^{\text{ref}})$	Reference temp. for CP i	Average difference in temperature between SDM and direct FEM	Maximum difference in temperature between SDM and direct FEM
Symbol	m_L	u_i^{ref}	u	
Unit			$^\circ\text{C}$	
SDM	10	u_i^G (Eq. (46))	0.2271079	0.3873859
SDM	10	Average (Eq. (48))	0.2271081	0.3873862
SDM	20	u_i^G (Eq. (46))	0.2273832	0.3876047
SDM	20	Average (Eq. (48))	0.2273832	0.3876047

imaginary material. In this example, the material nonlinearity (Fig. 11) was stronger than the real fiber composite (Fig. 10). Similar to the real material, m_L and the reference temperature calculation did not have a significant effect on the accuracy of the SDM. The four SDM results in Table 3 have almost the same accuracy. However, the temperature differences are much larger compared with those of the real material. Figures 17(a)–(c) contain the profiles of the temperature (u) and its gradients ($\partial u/\partial x_1$ and $\partial u/\partial x_2$), for the first imaginary material. Although the distributions calculated by the SDM (925 CPs, $m_L = 20$, and reference temperatures calculated using Eq. (48)) roughly agree with the FEM results, they are partially discontinuous. As stated in the abstract and introduction, the proposed SDM is a simple nonlinear solver that does not use an iterative multiscale analysis at every convergence calculation. In the SDM, the microscopic unit cells are analyzed using a linear FEM without a nonlinear solver. However, the SDM cannot give an exact solution for any problem that has strong nonlinearities. Consider that there was a temperature change of 10 °C over the entire structure, it would depend upon the particular example whether an average temperature error of 0.083 °C is significant (non-negligible or negligible). Although we cannot guarantee an exact solution at all times, the SDM analysis would be practical as a quick method for solving a nonlinear problem with a certain degree of accuracy.

Table 4 and Fig. 18 present the analytical results for the second imaginary structure, which had the largest material nonlinearities of the numerical examples (Fig. 12). The average temperature error was 0.227 °C, i.e., 2.27 % of the temperature change over the entire structure. Additionally, there were large differences in the temperature gradient contours from the SDM and FEM. Consequently, we do not believe that the SDM was sufficiently accurate for this material. Further improvements would be required to solve problems such as this, which we will investigate in the future.

4. Conclusions

Previous manuscripts demonstrated that the SDM multiscale numerical technique, can accurately solve linear steady-state heat conduction (Suzuki and Soga, 2016) and elasticity (Suzuki, 2016a, 2016b) problems. This study presented a simple nonlinear solver for the SDM analysis of heterogeneous materials, and investigated the practicalities of applying the SDM scheme to nonlinear stationary temperature fields. The proposed method easily approximated solutions to elliptic boundary value problems, and did not use an iterative multiscale analysis at every convergence

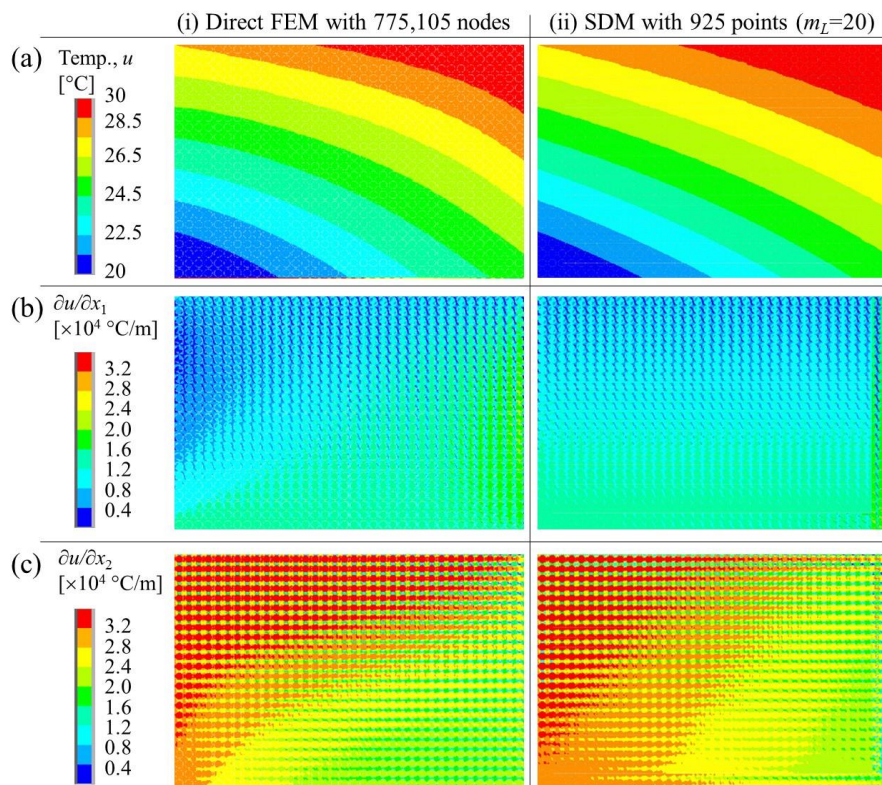


Fig. 18 Calculated contours for Imaginary Heterogeneous Material 2 (shown in Fig. 12): (a) temperature, u ; and temperature gradients (b) $\partial u/\partial x_1$ and (c) $\partial u/\partial x_2$.

calculation.

For a real heterogeneous material (graphite fiber composite) with a thermal conductivity that does not strongly depend on temperature, the SDM with only 925 points provided as an accurate solution as the ordinary FEM using hundreds of thousands of nodes. The average difference in the temperatures from the SDM and direct FEM was 1.18×10^{-3} °C, for a temperature change of 10 °C (from 20 to 30 °C) over the entire structure. We did not see differences in the contours of the temperatures and gradients derived by the two methods. The computational time of the direct FEM was over 44 times that of the SDM.

However, the SDM produced errors when analyzing imaginary heterogeneous materials that had strong material nonlinearities. If the thermal conductivity was more dependent on temperature, the error was larger. For an imaginary material where the inclusion's conductivity at 20 °C was ten times larger than that at 30 °C, the average temperature difference between the SDM and FEM solutions was 2.27% of the temperature change over the entire structure. The SDM cannot sufficiently solve a problem with a strong nonlinearity like this. Although we cannot guarantee an exact solution at all times, the SDM would be useful as a quick nonlinear method that gives a solution with a certain degree of accuracy.

Note that the current version of the SDM does not interactively couple the multiple scales. In other words, it does not conduct an iterative multiscale analysis at every convergence calculation, and only conducts a linear analysis of the local domain for the microscopic analysis. Therefore, we could improve the analytical precision by conducting both local and global analyses for every convergence calculation. We will develop this “iterative” SDM scheme in the future.

In addition, for other kinds of partial differential equations (such as stress equilibrium equations like the nonlinear elasto-plastic problem), the SDM requires additional extensions. We will develop these functionalities in the future. Unlike a simulated scalar field such as a temperature field, a two- or three-dimensional elasto-plastic field is a vector field and seems to be more difficult to solve than the scalar field. Depending on degree of the nonlinearity, a sufficiently accurate analysis of nonlinear elasto-plasticity employing the SDM without an iterative analysis, which is proposed in this article, would be impossible in some cases. Conversely, there is a substantial possibility that the SDM with iterative calculations, i.e., the “iterative” SDM scheme described in the previous paragraph, gives a practically precise solution to the nonlinear problem. It is noted that the “iterative” SDM algorithm is more computationally expensive than the SDM without the iterative calculations.

References

- Abdulle, A., Engquist, B., Finite element heterogeneous multiscale methods with near optimal computational complexity, *Multiscale modeling and simulation*, Vol. 6, No. 4 (2007), pp.1059–1084.
- Chua, J., Efendiev, Y., Ginting, V., Hou, T.Y., Flow based oversampling technique for multiscale finite element methods, *Advances in Water Resources*, Vol. 31, No. 4 (2008), pp.599–608.
- Efendiev, Y., Galvis, J., Li, G., Presho, M., Generalized multiscale finite element methods. Oversampling strategies. *arXiv(mathe NA) 2013; arXiv:1304.4888*.
- Fish, J., Markolefas, S., Guttal, R., Nayak P., On adaptive multilevel superposition of finite element meshes, *Applied Numerical Mathematics*, Vol. 14, (1994), pp.135–164.
- Fish, J., Yu, Q., Multiscale damage modelling for composite materials: theory and computational framework, *International Journal of Numerical Method in Engineering*, Vol. 52, No. 1–2 (2001), pp.161–191.
- Henning, P., Peterseim, D., Oversampling for the multiscale finite element method. *arXiv(mathe NA) 2013; arXiv:1211.5954*.
- Hesthaven, J. S., Zhang S., Zhu, X., High-Order Multiscale Finite Element Method for Elliptic Problems, *Multiscale Modeling and Simulation*, Vol. 12, No. 2 (2014), pp.650–666.
- Ilic, S., Hackl, K., Application of the multiscale FEM to the modeling of nonlinear multiphase materials, *Journal of theoretical and applied mechanics*, Vol. 47, No. 3 (2009), pp.537–551.
- Jenny, P., Lee, S. H., Tchelepi, H. A., Multi-scale finite-volume method for elliptic problems in subsurface flow simulation, *Journal of Computational Physics*, Vol. 187, No. 1 (2003), pp.47–67.
- Jenny, P., Lee, S. H., Tchelepi, H. A., Adaptive multiscale finite-volume method for multiphase flow and transport in porous media, *Multiscale Modeling and Simulation*, Vol. 3, No. 1 (2004), pp.50–64.
- Jiang, W. G., Hallett, S. R., Wisnom, M. R., Development of domain superposition technique for the modelling of

- woven fabric composites, *Computational Methods in Applied Sciences*, Vol. 10, (2008), pp.281–291.
- Kaczmarczyk, L., Pearce, C. J., Bićanić, N., Studies of microstructural size effect and higher-order deformation in second-order computational homogenization, *Computers & Structures*, Vol. 88, No. 23–24 (2010), pp.1383–1390.
- Lunati, I., Jenny, P., Multiscale finite-volume method for compressible multiphase flow in porous media, *Journal of Computational Physics*, Vol. 216, No. 2 (2006), pp.616–636.
- Liu, W. K., Hao, S., Belytschko, T., Multi-scale Methods, *International Journal of Numerical Method in Engineering*, Vol. 47, (2000), pp.1343–1361.
- Markovic, D., Niekamp, R., Ibrahimbegovic, A., Matthies, H. G., Taylor, R. L., Multi-scale modeling of heterogeneous structures with inelastic constitutive behaviour: Part I-physical and mathematical aspects, *Engineering Computations*, Vol. 25, (2005), pp.664–683.
- McDowell, D. L., A perspective on trends in multiscale plasticity, *International Journal of Plasticity*, Vol. 26, (2010), pp.1280–1309.
- Park, J. W., Hwang, J. W., Kim, Y. H., Efficient finite element analysis using mesh superposition technique, *Finite Elements in Analysis and Design*, Vol. 39, No. 7 (2003), pp.619–638.
- Rojek, J., Oñate, E., Multiscale analysis using a coupled discrete/finite element model, *Interaction and Multiscale Mechanics*, Vol. 1, No. 1 (2007), pp.1–31.
- Roters, F., Eisenlohr, P., Hantcherli, L., Tjahjanto, D. D., Bieler, T. R., Raabe, D., Overview of constitutive laws, kinematics, homogenization and multiscale methods in crystal plasticity finite-element modeling: Theory, experiments, applications, *Acta Materialia*, Vol. 58, (2010), pp.1152–1211.
- Suzuki, Y., Soga, K., Seamless domain method: a meshfree multiscale numerical analysis, *International Journal of Numerical Method in Engineering*, (2016), in publish.
- Suzuki, Y., Multiscale seamless-domain method for linear elastic analysis of heterogeneous materials, *International Journal of Numerical Method in Engineering*, Vol. 105, No. 8 (2016a), pp.563–598.
- Suzuki, Y., Three-scale modeling of laminated structures employing the seamless-domain method, *Composite structures*, (2016b), in publish.
- Terada, K., Hori, M., Kyoya, T., Kikuchi. Simulation of the multi-scale convergence in computational homogenization approaches, *International Journal of Solids and Structures*, Vol. 37 (2000), pp.2285–2311.
- Zhang, H. W., Wu, J. K., Lv, J., A new multiscale computational method for elasto-plastic analysis of heterogeneous materials, *Computational Mechanics*, Vol. 49 (2012), pp.149–169.
- Zhang, X., Fujiwara, S., Fujii, M., Measurements of thermal conductivity and electrical conductivity of a single carbon fiber, *International Journal of Thermophysics*, Vol. 21, No. 4 (2000), pp.965–980.

Appendix: how to apply the proposed SDM scheme to actual composite components and products

Actual composite components and products are much larger compared to the heterogeneous structures that are provided in Section 3: Numerical examples. Here we illustrate two methods to apply the multiscale SDM scheme to an analysis of a real-size composite structure having enormous number of inclusions.

Three-scale modeling employing both the SDM and the homogenization method (Suzuki, 2016b)

In previous work (Suzuki, 2016b), we presented three-scale modeling of laminated structures employing both the SDM and the homogenization method and applied it to linear problems such as elastic deformation and heat conduction. Figure 19 illustrates the procedure of the proposed three-scale analysis.

First, we homogenize heterogeneous constitutive layers to obtain homogenized material properties (Fig. 19(a)(b)) based on the so-called homogenization theory (Terada, et al., 2000, Kaczmarczyk, et al., 2010). This process converts each heterogeneous layer into an orthotropic homogeneous layer.

Second, we conduct mesoscopic analysis of part of the multilayered structure employing an FEM solver as shown in Fig. 19(c). Each layer of the laminate behaves as a homogeneous continuum. This process is equivalent to the SDM's

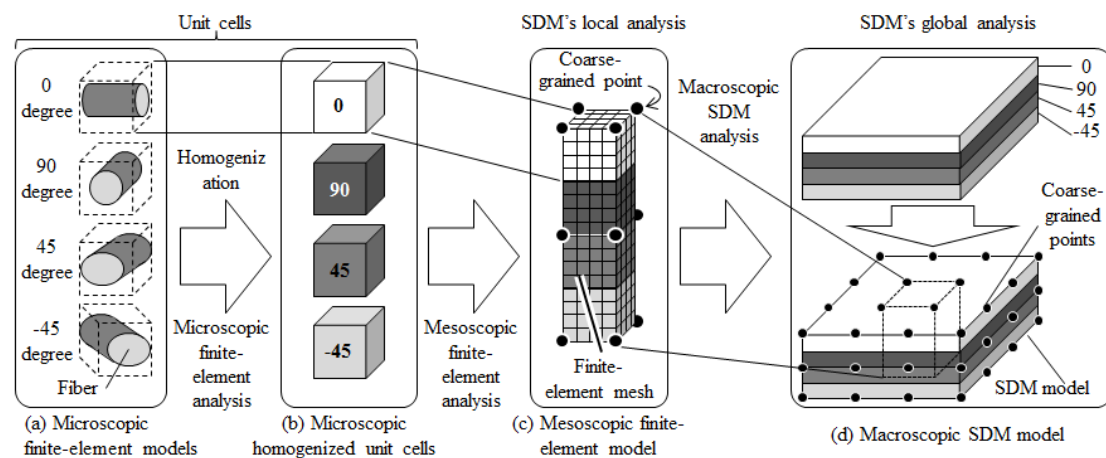


Fig. 19 Schematic of procedure of a three-scale analysis of an actual laminated composite structure using the SDM, homogenization technique, and finite-element method: (a) microscopic unit cells; (b) microscopic homogenized unit cells; (c) a mesoscopic partial laminate model; and (d) macroscopic laminated structure (Suzuki, 2016).

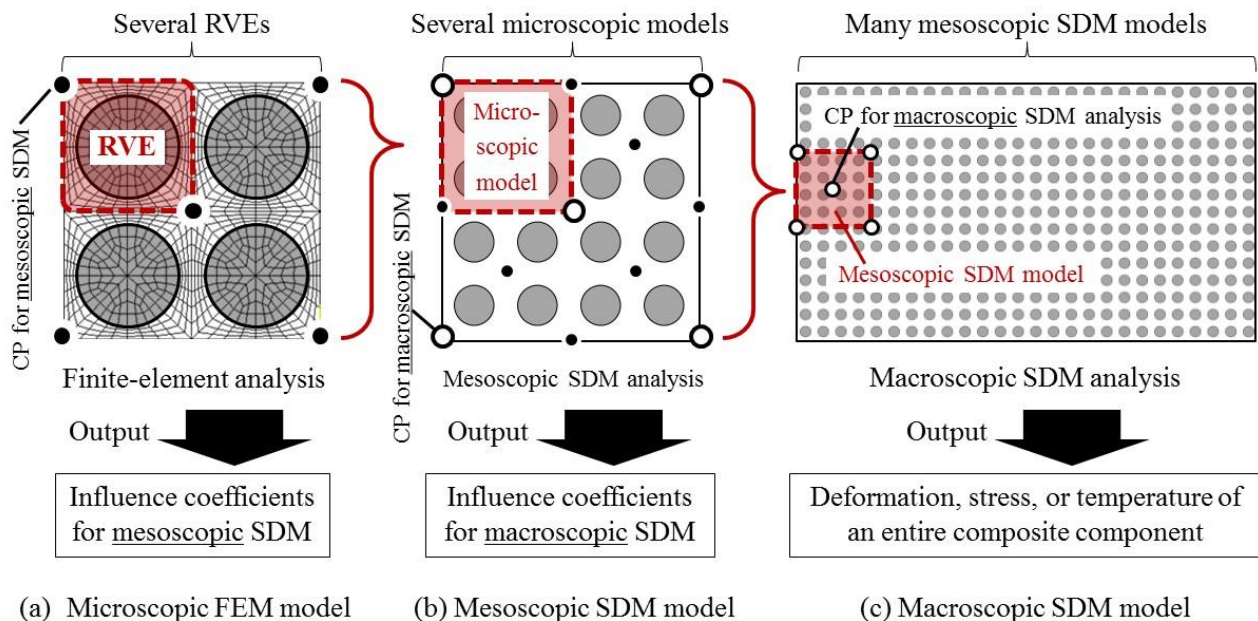


Fig. 20 Schematic of the procedure of a three-scale analysis of an actual composite structure without homogenization technique: (a) microscopic finite-element model consisting of several unit cells; (b) microscopic SDM model composed of several microscopic domains; and (c) macroscopic global domain, i.e., an entire structure.

local analysis (Subsection 2.4.1) and generates interpolation functions like $\mathbf{N}^{Ri}$ in Eq. (34) and influence coefficients like $\mathbf{a}^{Ri}$ in Eq. (35) for the mesoscopic part of the laminate.

Finally, we move on to macroscopic analysis of the entire laminated structure (Fig. 19(d)), i.e., the SDM's global analysis. The entire structure, i.e., global domain is represented by only coarse-grained points (CPs). Relation among dependent-variable values at neighboring CPs is determined by $\mathbf{a}^{Ri}$.

Consequently, the three-scale modeling technique achieves a computationally inexpensive and substantially precise analysis of an actual laminated composite. The technique takes into consideration the structural complexity which results from the laminate consisting of strongly anisotropic heterogeneous plies. In all numerical examples in the paper (Suzuki, 2016b), the three-scale models represented by a small number of points provided solutions as accurate as those of standard finite-solid-element models having many nodes.

Three-scale modeling based on two-step SDM

Unlike the three-scale modeling illustrated in the previous subsection, we present a different three-scale analysis without the homogenization method, which is composed of

- microscopic analysis of an FEM model consisting of several RVEs (Fig. 20(a));
- mesoscopic analysis of an SDM model (i.e., part of an actual composite component) consisting of several microscopic models (Fig. 20(b)); and
- macroscopic analysis of an SDM model (i.e., the entire component) consisting of many mesoscopic models (Fig. 20(c)).

The calculation procedure of the microscopic analysis is the same as the local analysis that is proposed in this article and explained in Subsection 2.4.1. This analysis generates influence coefficients for the microscopic model that are used in the subsequent mesoscopic analysis.

The mesoscopic analysis can be conducted in the same way as the SDM global analysis that is presented in this paper (Subsection 2.4.2). In the analysis, dependent-variable value at a CP (i.e., black circle in Fig. 20(a)(b)) is determined referring to variables at its surrounding CPs. The analysis gives influence coefficients for the mesoscopic model. The coefficients are used to determine relation among neighboring CPs (white circles in Fig. 20(b)(c)) in the subsequent macroscopic analysis.

We analyze the entire composite component (macroscopic model) employing the SDM methodology and the influence coefficients obtained in the mesoscopic analysis. As shown in Fig.20(c), the macroscopic model is represented by only white CPs. The fine finite-element mesh (Fig. 20(a)) and the black CPs (Fig. 20(b)) do not exist.

Compared with the two-scale SDM analysis presented in this article, the three-scale SDM analysis that is described in this subsection would be able to reduce total computational cost and to analyze a larger heterogeneous structure. The three-scale modeling scheme can be extended to four- or more-scale analysis and lead to further improvement in computational efficiency.

Nomenclature

Symbol	Explanation
$\#^{-1}$	Inverse of #
$\#^T$	Transpose of #
$\Omega_G \subset \mathbf{R}^d$	d -dimensional domain for global analysis
$\Omega_{Li} \subset \mathbf{R}^d$	i -th d -dimensional local domain
$\Omega_{Ri} \subset \Omega_{Li}$	i -th region of influence
Γ_G	Boundary of the global domain
Γ_{Li}	Boundary of the i -th local domain
Γ_{Ri}	Boundary of the i -th region of influence
$\mathbf{a}^{Ri} \in \mathbf{R}^{1 \times m_i}$	Influence coefficient matrix, which estimates a dependent variable for a CP in the center of Ω_{Ri} ($u_{\text{explained}}^{Ri}$) referring to $\mathbf{u}_{\text{predictor}}^{Ri}$, having no dependency on the dependent variable
$\mathbf{a}^{Ri}(u^{\text{ref}}) \in \mathbf{R}^{1 \times m_i}$	Influence coefficient matrix at temperature u^{ref} for Ω_{Ri}
$\mathbf{a}^G \in \mathbf{R}^{n \times n}$	n -by- n matrix established by assembling all the influence coefficient matrices, $\mathbf{a}^{Ri}(u^{\text{ref}})$
$\mathbf{A}^{Ri}(\mathbf{x}) \in \mathbf{R}^{1 \times m_i}$	1-by- m_i matrix for calculating $u(\mathbf{x})$ from $\mathbf{u}^{Li}$
$\mathbf{A}_{\text{predictor}}^{Ri} \in \mathbf{R}^{m_i \times m_i}$	m_i -by- m_i matrix for calculating $\mathbf{u}_{\text{predictor}}^{Ri}$ from $\mathbf{u}^{Li}$
$d \in \{1, \dots, 3\}$	Dimension of domain
m_i	Number of coarse-grained points (CPs) in the i -th region of influence
m_L	Number iterations of the local analysis, i.e., number of $\mathbf{a}^{Ri}(u^{\text{ref}})$
n	Number of CPs in a global domain
$\mathbf{N}^{Ri}(\mathbf{x}) \in \mathbf{R}^{1 \times m_i}$	1-by- m_i matrix, which interpolates dependent-variable distribution in Ω_{Ri} referring to $\mathbf{u}_{\text{predictor}}^{Ri}$, having no dependency on the dependent variable
$\mathbf{N}^{Ri}(\mathbf{x}, u^{\text{ref}}) \in \mathbf{R}^{1 \times m_i}$	Interpolating function matrix for $\mathbf{u}_{\text{predictor}}^{Ri}$ at temperature u^{ref} for Ω_{Ri}
$\mathbf{R}$	Set of all real numbers
$u(\mathbf{x}) \in \mathbf{R}$	Dependent variable at point $\mathbf{x}$
$\mathbf{u}^G \in \mathbf{R}^n$	Dependent variable of CPs in a global domain (Ω_G)
$\mathbf{u}^{Li} \in \mathbf{R}^{m_i}$	Dependent variable for all CPs near the i -th local domain's boundary
$u_{\text{explained}}^{Ri} \in \mathbf{R}$	Dependent variable for a CP in the center of Ω_{Ri} that is calculated referring to $\mathbf{u}_{\text{predictor}}^{Ri}$ using $\mathbf{a}^{Ri}$
$\mathbf{u}_{\text{predictor}}^{Ri} \in \mathbf{R}^{m_i}$	Dependent variable for CPs in Ω_{Ri} that predicts $u_{\text{explained}}^{Ri}$ using $\mathbf{a}^{Ri}$
u_i^{ref}	Reference temperature for CP i
$\mathbf{x} \in \mathbf{R}^d$	Position vector