

Stresses around an eccentric hole in an infinite strip subjected to in-plane bending

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Abstract

In this paper, we present an analytical solution for an infinite strip having an eccentric circular hole when the strip is subjected to in-plane bending at infinity. The analysis is based on the Papkovitch-Neuber displacement potentials and the solution is obtained by a proper combination of harmonic function in integral forms and infinite series. The boundary conditions on both sides of the strip and around the hole are satisfied using the relations between the Cartesian and polar harmonics. The numerical results obtained are compared with those of FEM. A detailed stresses around the eccentric hole are illustrated for the various size of the eccentricity and the hole radius.

Key words : Elasticity, Stress concentration, Infinite strip, Eccentric circular hole, In-plane bending, Analytical solution

1. Introduction

The presence of defects in a material gives rise to stress concentrations that may cause either yielding or cracking. Therefore, it is crucially important to know the stresses around the defects in engineering design (Savin, 1960, Nakahara et al., 2001). Until now, the stress concentration problems of a strip with a hole or an inclusion have been addressed by many researchers (e.g., Howland, 1930, Ishida, 1953, Ling, 1957, Itou, 1976, Theotokoglou et al., 1986, Lee et al., 1992, Horibe et al., 2008, Luo 2012). The stress concentration factor tends to increase when an elastic body have an asymmetrically, or, eccentrically located defect. Thus, the stress concentration caused by the eccentric defect must also be clarified to retain the sufficient strength. Few researchers (Tsuchida et al. 1976, Tsuchida et al., 1978, Kawakami et al., 1980, Kawakami et al., 1981, Horibe et al., 2007, Luo et al., 2012) have discussed for these problems since eccentricity makes the analysis more complicated. Such asymmetric problems are practically important ones in engineering design and further analysis is expected. In this paper, a solution for an elastic strip having an eccentrically located circular hole is presented when the strip is subjected to in-plane bending at infinity. This problem was already solved by Ling C. B. and Ishida M. using the Airy stress function, but a limited number of numerical results is shown in the papers. Present analysis is developed on the basis of Papkovitch-Neuber displacement potential approach. The solution is deduced using the simple forms of Cartesian and cylindrical harmonics in an infinite series and an infinite integral form. The boundary conditions for the strip are satisfied using the relations between the Cartesian and the cylindrical harmonics. Several numerical examples are given to clarify the stress concentration due to the eccentrically located circular hole.

2. Method of Solution

2.1. Formulation

Consider the infinite elastic strip having an eccentrically located hole shown in Fig.1. Let the origin of coordinates be at the center of the hole and the x -axis be normal to the boundary of the strip. Figure 1 shows the Cartesian coordinates (x, y) and the cylindrical coordinates (r, θ) . The half width of the strip is taken as a unit length.

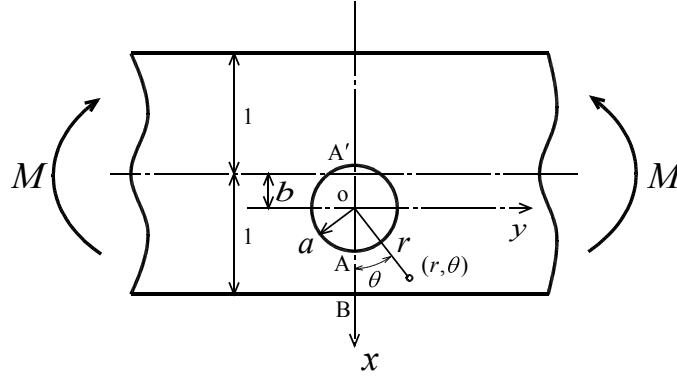


Fig. 1 Coordinates system

Therefore, without any loss in generality, x, y and r are regarded as dimensionless quantities by referring to a typical length of the half-width of the strip and we denote the radius of the hole by $r = a$, both sides of the strip by $x = 1 - b$ and $x = -1 - b$, where b is the eccentricity. An in-plane bending moment M is applied at infinity as shown in Fig.1. The governing differential equations, representing equilibrium equations in terms of displacements (u, v) , are (Nakahara et al., 2001)

$$\begin{cases} \nabla^2 u + \frac{2}{\kappa - 1} \cdot \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \frac{X}{G} = 0, \\ \nabla^2 v + \frac{2}{\kappa - 1} \cdot \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \frac{Y}{G} = 0, \end{cases} \quad (1)$$

where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is Laplacian and $\kappa = 3 - 4\nu$ (for plane strain) or $\kappa = (3 - \nu)/(1 + \nu)$ (for plane stress), where ν is Poisson's ratio. Following Papkovitch and Neuber, the displacements can be expressed in terms of the harmonic displacement potentials φ_0 and φ_1 as (Nakahara et al., 2001)

$$\begin{cases} 2Gu = \frac{\partial}{\partial x}(\varphi_0 + x\varphi_1) - (\kappa + 1)\varphi_1, \\ 2Gv = \frac{\partial}{\partial y}(\varphi_0 + x\varphi_1), \end{cases} \quad (2)$$

where G is the shear modulus, $\nabla^2 \varphi_0 = 0$ and $\nabla^2 \varphi_1 = 0$. The boundary conditions of the present problem are as follows:

(i) Normal and shear stress must be free on both sides of the strip $x = 1 - b$ and $x = -1 - b$;

$$\begin{aligned} \left(\frac{\sigma_x}{k} \right)_{x=1-b} &= \left(\frac{\tau_{xy}}{k} \right)_{x=1-b} = 0, \\ \left(\frac{\sigma_x}{k} \right)_{x=-1-b} &= \left(\frac{\tau_{xy}}{k} \right)_{x=-1-b} = 0, \end{aligned} \quad (3)$$

where $k = M/(2^2/6) = 3M/2$, which means the maximum bending stress arising in a strip with the height 2.

(ii) Normal and shear stress around the hole $r = a$ must be free;

$$\left(\frac{\sigma_r}{k} \right)_{r=a} = 0, \quad \left(\frac{\tau_{r\theta}}{k} \right)_{r=a} = 0. \quad (4)$$

(iii) Normal stress in y direction must be equal to the external traction kx and normal stress in x direction and shear stress must vanish at infinity;

$$\sigma_y = kx, \quad \sigma_x = 0, \quad \tau_{xy} = 0. \quad (5)$$

In deriving the required displacement potentials, we should consider that the elastic body consists of the common region of two simply connected domains, i.e., the domain of a strip and that of an infinite body excluding a hole, and we make use of the two groups of potentials, those which have no singularities in the strip and those which have singularities at the center of the hole. Such potentials are expressed by the following Cartesian harmonics and the cylindrical harmonics:

$$[I] \begin{cases} \varphi_0 = -\frac{(\kappa-1)}{24}k(x^3 - 3xy^2) = -\frac{(\kappa-1)}{24}kr^3 \cos 3\theta, \\ \varphi_1 = -\frac{k}{4}(x^2 - y^2) = -\frac{k}{4}r^2 \cos 2\theta. \end{cases} \quad (6)$$

$$[II] \begin{cases} \varphi_0 = k \sum_{m=1}^{\infty} A_m \frac{\cos m\theta}{r^m}, \quad \varphi_1 = k \sum_{m=1}^{\infty} B_m \frac{\cos(m+1)\theta}{r^{m+1}}. \end{cases} \quad (7)$$

$$[III] \begin{cases} \varphi_0 = k \int_0^{\infty} \left[\psi_1(\lambda) \cosh \lambda(x+b) + \psi_2(\lambda) \sinh \lambda(x+b) \right] \cos \lambda y \, d\lambda, \\ \varphi_1 = k \int_0^{\infty} \lambda \left[\psi_3(\lambda) \cosh \lambda(x+b) + \psi_4(\lambda) \sinh \lambda(x+b) \right] \cos \lambda y \, d\lambda, \end{cases} \quad (8)$$

where A_m , B_m and $\psi_1(\lambda)$, $\psi_2(\lambda)$, $\psi_3(\lambda)$, $\psi_4(\lambda)$ are unknown constants and unknown functions, which are determined from the boundary conditions, respectively. Considering the x -axis symmetry and y -axis asymmetry of the stress field, we have selected the potentials such as the right-hand side of Eqs.(7) and (8). The displacements and stresses derived from Eq.(6) are expressed as

$$\begin{aligned} 2Gu_x &= -\frac{1}{8}k \{ (3-\kappa)x^2 + (1+\kappa)y^2 \}, \quad 2Gv_y = \frac{1}{4}k(\kappa+1)xy, \\ \sigma_x &= 0, \quad \sigma_y = kx, \quad \tau_{xy} = 0. \end{aligned} \quad (9)$$

Thus, the potentials [I] yield the stress state that the strip is subjected to in-plane bending of M at infinity. Bending moment M is given so that a bending stress of a lower part of a strip becomes tensile. Therefore, the boundary condition (iii) is automatically satisfied, since the stress components derived from the potentials [II] and [III] vanish at infinity.

Complicated equations such as Eq.(7) or Eq.(8) are used throughout in the paper. To avoid the careless errors in manipulating those equations, we employed symbolic computation system Mathematica.

2.2. Boundary Conditions on Both Sides of the Strip

To satisfy the boundary conditions on both sides of the strip, we employ the following transformation formula of harmonic functions:

$$\left. \begin{aligned} \frac{\cos m\theta}{r^m} &= \frac{1}{(m-1)!} \int_0^{\infty} \lambda^{m-1} e^{-\lambda x} \cos \lambda y \, d\lambda \\ \frac{\sin m\theta}{r^m} &= \frac{1}{(m-1)!} \int_0^{\infty} \lambda^{m-1} e^{-\lambda x} \sin \lambda y \, d\lambda \end{aligned} \right\} (x > 0), \quad \left. \begin{aligned} \frac{\cos m\theta}{r^m} &= \frac{(-1)^m}{(m-1)!} \int_0^{\infty} \lambda^{m-1} e^{\lambda x} \cos \lambda y \, d\lambda \\ \frac{\sin m\theta}{r^m} &= \frac{(-1)^{m+1}}{(m-1)!} \int_0^{\infty} \lambda^{m-1} e^{\lambda x} \sin \lambda y \, d\lambda \end{aligned} \right\} (x < 0). \quad (10)$$

If we rewrite the displacement potential [II] into the sum of odd and even orders of m , i.e.,

$$[II] \begin{cases} \varphi_0 = k \sum_{m=1}^{\infty} A_m \frac{\cos m\theta}{r^m} = k \sum_{m=1}^{\infty} A_{2m} \frac{\cos 2m\theta}{r^{2m}} + k \sum_{m=0}^{\infty} A_{2m+1} \frac{\cos(2m+1)\theta}{r^{2m+1}}, \\ \varphi_1 = k \sum_{m=1}^{\infty} B_m \frac{\cos(m+1)\theta}{r^{m+1}} = k \sum_{m=1}^{\infty} B_{2m-1} \frac{\cos 2m\theta}{r^{2m}} + k \sum_{m=1}^{\infty} B_{2m} \frac{\cos(2m+1)\theta}{r^{2m+1}}, \end{cases} \quad (11)$$

we obtain displacement potentials [II] in the following Cartesian coordinates using the relation (10):

$$\begin{aligned}
 [\text{II} - 1\text{p}] \begin{cases} \varphi_0 = k \sum_{m=1}^{\infty} \frac{A_{2m}}{(2m-1)!} \int_0^{\infty} \lambda^{2m-1} e^{-\lambda x} \cos \lambda y \, d\lambda + k \sum_{m=0}^{\infty} \frac{A_{2m+1}}{(2m)!} \int_0^{\infty} \lambda^{2m} e^{-\lambda x} \cos \lambda y \, d\lambda, \\ \varphi_1 = k \sum_{m=1}^{\infty} \frac{B_{2m-1}}{(2m-1)!} \int_0^{\infty} \lambda^{2m-1} e^{-\lambda x} \cos \lambda y \, d\lambda + k \sum_{m=1}^{\infty} \frac{B_{2m}}{(2m)!} \int_0^{\infty} \lambda^{2m} e^{-\lambda x} \cos \lambda y \, d\lambda, \end{cases} \quad (x > 0) \\
 [\text{II} - 1\text{m}] \begin{cases} \varphi_0 = k \sum_{m=1}^{\infty} \frac{A_{2m}}{(2m-1)!} \int_0^{\infty} \lambda^{2m-1} e^{\lambda x} \cos \lambda y \, d\lambda - k \sum_{m=0}^{\infty} \frac{A_{2m+1}}{(2m)!} \int_0^{\infty} \lambda^{2m} e^{\lambda x} \cos \lambda y \, d\lambda, \\ \varphi_1 = k \sum_{m=1}^{\infty} \frac{B_{2m-1}}{(2m-1)!} \int_0^{\infty} \lambda^{2m-1} e^{\lambda x} \cos \lambda y \, d\lambda - k \sum_{m=1}^{\infty} \frac{B_{2m}}{(2m)!} \int_0^{\infty} \lambda^{2m} e^{\lambda x} \cos \lambda y \, d\lambda. \end{cases} \quad (x < 0)
 \end{aligned} \quad (12)$$

To satisfy the boundary conditions on both sides of the strip, we derive stress components from the potentials [I], [III] and [IV] as follows.

$$\begin{aligned}
 \left(\frac{\sigma_x}{k} \right)_{x=1-b} = \int_0^{\infty} \lambda^2 \left[\cosh \lambda \, \psi_1(\lambda) + \sinh \lambda \, \psi_2(\lambda) + \left\{ (1-b)\lambda \cosh \lambda - \frac{\kappa+1}{2} \sinh \lambda \right\} \psi_3(\lambda) \right. \\
 + \left\{ (1-b)\lambda \sinh \lambda - \frac{\kappa+1}{2} \cosh \lambda \right\} \psi_4(\lambda) + \sum_{m=1}^{\infty} \frac{\lambda^{2m-1}}{(2m-1)!} e^{-(1-b)\lambda} A_{2m} \\
 + \sum_{m=0}^{\infty} \frac{\lambda^{2m}}{(2m)!} e^{-(1-b)\lambda} A_{2m+1} + \sum_{m=1}^{\infty} \frac{\lambda^{2m-2}}{(2m-1)!} e^{-(1-b)\lambda} \left\{ (1-b)\lambda + \frac{\kappa+1}{2} \right\} B_{2m-1} \\
 \left. + \sum_{m=1}^{\infty} \frac{\lambda^{2m-1}}{(2m)!} e^{-(1-b)\lambda} \left\{ (1-b)\lambda + \frac{\kappa+1}{2} \right\} B_{2m} \right] \cos \lambda y \, d\lambda = 0, \quad (13)
 \end{aligned}$$

$$\begin{aligned}
 \left(\frac{\tau_{xy}}{k} \right)_{x=1-b} = \int_0^{\infty} \lambda^2 \left[-\sinh \lambda \, \psi_1(\lambda) - \cosh \lambda \, \psi_2(\lambda) - \left\{ (1-b)\lambda \sinh \lambda - \frac{\kappa-1}{2} \cosh \lambda \right\} \psi_3(\lambda) \right. \\
 - \left\{ (1-b)\lambda \cosh \lambda - \frac{\kappa-1}{2} \sinh \lambda \right\} \psi_4(\lambda) + \sum_{m=1}^{\infty} \frac{\lambda^{2m-1}}{(2m-1)!} e^{-(1-b)\lambda} A_{2m} \\
 + \sum_{m=0}^{\infty} \frac{\lambda^{2m}}{(2m)!} e^{-(1-b)\lambda} A_{2m+1} + \sum_{m=1}^{\infty} \frac{\lambda^{2m-2}}{(2m-1)!} e^{-(1-b)\lambda} \left\{ (1-b)\lambda + \frac{\kappa-1}{2} \right\} B_{2m-1} \\
 \left. + \sum_{m=1}^{\infty} \frac{\lambda^{2m-1}}{(2m)!} e^{-(1-b)\lambda} \left\{ (1-b)\lambda + \frac{\kappa-1}{2} \right\} B_{2m} \right] \sin \lambda y \, d\lambda = 0, \quad (14)
 \end{aligned}$$

$$\begin{aligned}
 \left(\frac{\sigma_x}{k} \right)_{x=-1-b} = \int_0^{\infty} \lambda^2 \left[\cosh \lambda \, \psi_1(\lambda) - \sinh \lambda \, \psi_2(\lambda) + \left\{ -(1+b)\lambda \cosh \lambda + \frac{\kappa+1}{2} \sinh \lambda \right\} \psi_3(\lambda) \right. \\
 + \left\{ (1+b)\lambda \sinh \lambda - \frac{\kappa+1}{2} \cosh \lambda \right\} \psi_4(\lambda) + \sum_{m=1}^{\infty} \frac{\lambda^{2m-1}}{(2m-1)!} e^{-(1+b)\lambda} A_{2m} \\
 - \sum_{m=0}^{\infty} \frac{\lambda^{2m}}{(2m)!} e^{-(1+b)\lambda} A_{2m+1} - \sum_{m=1}^{\infty} \frac{\lambda^{2m-2}}{(2m-1)!} e^{-(1+b)\lambda} \left\{ (1+b)\lambda + \frac{\kappa+1}{2} \right\} B_{2m-1} \\
 \left. + \sum_{m=1}^{\infty} \frac{\lambda^{2m-1}}{(2m)!} e^{-(1+b)\lambda} \left\{ (1+b)\lambda + \frac{\kappa+1}{2} \right\} B_{2m} \right] \cos \lambda y \, d\lambda = 0, \quad (15)
 \end{aligned}$$

$$\begin{aligned}
 \left(\frac{\tau_{xy}}{k} \right)_{x=-1-b} = \int_0^{\infty} \lambda^2 \left[\sinh \lambda \, \psi_1(\lambda) - \cosh \lambda \, \psi_2(\lambda) - \left\{ (1+b)\lambda \sinh \lambda - \frac{\kappa-1}{2} \cosh \lambda \right\} \psi_3(\lambda) \right. \\
 + \left\{ (1+b)\lambda \cosh \lambda - \frac{\kappa-1}{2} \sinh \lambda \right\} \psi_4(\lambda) - \sum_{m=1}^{\infty} \frac{\lambda^{2m-1}}{(2m-1)!} e^{-(1+b)\lambda} A_{2m} \\
 + \sum_{m=0}^{\infty} \frac{\lambda^{2m}}{(2m)!} e^{-(1+b)\lambda} A_{2m+1} + \sum_{m=1}^{\infty} \frac{\lambda^{2m-2}}{(2m-1)!} e^{-(1+b)\lambda} \left\{ (1+b)\lambda + \frac{\kappa-1}{2} \right\} B_{2m-1} \\
 \left. - \sum_{m=0}^{\infty} \frac{\lambda^{2m-1}}{(2m)!} e^{-(1+b)\lambda} \left\{ (1+b)\lambda + \frac{\kappa-1}{2} \right\} B_{2m} \right] \sin \lambda y \, d\lambda = 0. \quad (16)
 \end{aligned}$$

Applying the inverse Fourier transform to Eqs.(13)-(16), we determine the unknown functions $\psi_1(\lambda)$, $\psi_2(\lambda)$, $\psi_3(\lambda)$ and $\psi_4(\lambda)$ to be

$$\begin{aligned}\psi_1(\lambda) = & \sum_{m=1}^{\infty} \frac{\lambda^{2m-1}}{2(2m-1)!} e^{-(1+b)\lambda} \left[\frac{(1+e^{2b\lambda})(e^{-\lambda} + (\kappa-2\lambda)e^{\lambda})}{2\lambda + \sinh 2\lambda} - \frac{2b\lambda e^{\lambda}(-1+e^{2b\lambda})}{2\lambda - \sinh 2\lambda} \right] A_{2m} \\ & + \sum_{m=0}^{\infty} \frac{\lambda^{2m} e^{-(1+b)\lambda}}{2(2m)!} \left[\frac{(-1+e^{2b\lambda})(e^{-\lambda} + (\kappa-2\lambda)e^{\lambda})}{2\lambda + \sinh 2\lambda} - \frac{2b\lambda e^{\lambda}(1+e^{2b\lambda})}{2\lambda - \sinh 2\lambda} \right] A_{2m+1} \\ & + \sum_{m=1}^{\infty} \frac{\lambda^{2m-2}}{(2m-1)!(4\lambda^2 - \sinh^2 2\lambda)} \left[-4b\lambda^2(\kappa + \cosh 2\lambda) \cosh b\lambda \right. \\ & \quad \left. + \left\{ \lambda(\kappa^2 - 1 - 4(1-b^2)\lambda^2) + (1 - \kappa^2 + 4(1+b^2)\lambda^2) \cosh \lambda \sinh \lambda \right\} \sinh b\lambda \right] B_{2m-1} \\ & + \sum_{m=1}^{\infty} \frac{\lambda^{2m-1}}{(2m)!(4\lambda^2 - \sinh^2 2\lambda)} \left[\left\{ \lambda(\kappa^2 - 1 - 4(1-b^2)\lambda^2) \right. \right. \\ & \quad \left. \left. + (1 - \kappa^2 + 4(1+b^2)\lambda^2) \cosh \lambda \sinh \lambda \right\} \cosh b\lambda - 4b\lambda^2(\kappa + \cosh 2\lambda) \sinh b\lambda \right] B_{2m},\end{aligned}\quad (17)$$

$$\begin{aligned}\psi_2(\lambda) = & \sum_{m=1}^{\infty} \frac{\lambda^{2m-1}}{2(2m-1)!(4\lambda^2 - \sinh^2 2\lambda)} \left[8b\lambda(\lambda - \cosh \lambda \sinh \lambda) \cosh b\lambda \right. \\ & \quad \left. - 2e^{-\lambda} \left\{ (\kappa - 1 - 2\lambda) \cosh \lambda + (\kappa + 1 - 2\lambda) \sinh \lambda \right\} (2\lambda + \sinh 2\lambda) \sinh b\lambda \right] A_{2m} \\ & - \sum_{m=0}^{\infty} \frac{\lambda^{2m}}{4(2m)!(4\lambda^2 - \sinh^2 2\lambda)} \left[e^{-(4+b)\lambda} (1 + e^{2b\lambda}) \right. \\ & \quad \left. - 2(1 - 4\kappa\lambda + 8\lambda^2 + 4\lambda \cosh 2\lambda - 2\kappa \sinh 2\lambda) \cosh b\lambda + 8b\lambda(-2\lambda + \sinh 2\lambda) \sinh b\lambda \right] A_{2m+1} \\ & + \sum_{m=1}^{\infty} \frac{\lambda^{2m-2}}{(2m-1)!(4\lambda^2 - \sinh^2 2\lambda)} \left[\left\{ (1 - \kappa^2)\lambda + 4(1-b^2)\lambda^3 \right. \right. \\ & \quad \left. \left. + (1 - \kappa^2 + 4(1+b^2)\lambda^2) \sinh \lambda \cosh \lambda \right\} \cosh b\lambda + 4b\lambda^2(\kappa - \cosh 2\lambda) \sinh b\lambda \right] B_{2m-1} \\ & + \sum_{m=1}^{\infty} \frac{\lambda^{2m-1}}{(2m)!(4\lambda^2 - \sinh^2 2\lambda)} \left[4b\lambda^2(\kappa - \cosh 2\lambda) \cosh b\lambda \right. \\ & \quad \left. + \left\{ \lambda(1 - \kappa^2 + 4(1-b^2)\lambda^2) + (1 - \kappa^2 + 4(1+b^2)\lambda^2) \sinh \lambda \cosh \lambda \right\} \sinh b\lambda \right] B_{2m},\end{aligned}\quad (18)$$

$$\begin{aligned}\psi_3(\lambda) = & - \sum_{m=1}^{\infty} \frac{2\lambda^{2m-1}}{(2m-1)!(2\lambda - \sinh 2\lambda)} \sinh b\lambda A_{2m} - \sum_{m=0}^{\infty} \frac{2\lambda^{2m}}{(2m)!(2\lambda - \sinh 2\lambda)} \cosh b\lambda A_{2m+1} \\ & + \sum_{m=1}^{\infty} \frac{\lambda^{2m-2}}{(2m-1)!(2\lambda - \sinh 2\lambda)} \left[- (e^{-2\lambda} + \kappa + 2\lambda) \cosh b\lambda + 2b\lambda \sinh b\lambda \right] B_{2m-1} \\ & + \sum_{m=1}^{\infty} \frac{\lambda^{2m-1}}{(2m)!(2\lambda - \sinh 2\lambda)} \left[2b\lambda \cosh b\lambda - (e^{-2\lambda} + \kappa + 2\lambda) \sinh b\lambda \right] B_{2m},\end{aligned}\quad (19)$$

$$\begin{aligned}
\psi_4(\lambda) = & \sum_{m=1}^{\infty} \frac{2\lambda^{2m-1}}{(2m-1)!(2\lambda + \sinh 2\lambda)} \cosh b\lambda A_{2m} + \sum_{m=0}^{\infty} \frac{2\lambda^{2m}}{(2m)!(2\lambda + \sinh 2\lambda)} \sinh b\lambda A_{2m+1} \\
& + \sum_{m=1}^{\infty} \frac{\lambda^{2m-2}}{(2m-1)!(2\lambda + \sinh 2\lambda)} \left[-2b\lambda \cosh b\lambda + (-e^{-2\lambda} + \kappa + 2\lambda) \sinh b\lambda \right] B_{2m-1} \\
& + \sum_{m=1}^{\infty} \frac{\lambda^{2m-1}}{(2m)!(2\lambda + \sinh 2\lambda)} \left[(-e^{-2\lambda} + \kappa + 2\lambda) \cosh b\lambda - 2b\lambda \sinh b\lambda \right] B_{2m}.
\end{aligned} \quad (20)$$

Equations (17)-(20) indicate the conditions of zero tractions on both sides of the strip. If these equations are satisfied, then boundary conditions, Eq.(3), will be satisfied for any values of the unknown constants A_m and B_m .

2.3. Boundary Conditions on Rim of Hole

Next, to satisfy the remaining boundary conditions on the rim of the hole given by Eq.(4), it is necessary to expand potential [II-1p] and [II-1m] in cylindrical coordinates. In view of the following relations of harmonics,

$$\begin{aligned}
\cosh \lambda(x+b) \cos \lambda y &= \cosh \lambda b \sum_{n=0}^{\infty} \frac{(\lambda r)^{2n}}{(2n)!} \cos 2n\theta + \sinh \lambda b \sum_{n=0}^{\infty} \frac{(\lambda r)^{2n+1}}{(2n+1)!} \cos(2n+1)\theta, \\
\sinh \lambda(x+b) \cos \lambda y &= \cosh \lambda b \sum_{n=0}^{\infty} \frac{(\lambda r)^{2n+1}}{(2n+1)!} \cos(2n+1)\theta + \sinh \lambda b \sum_{n=0}^{\infty} \frac{(\lambda r)^{2n}}{(2n)!} \cos 2n\theta,
\end{aligned} \quad (21)$$

the resulting expansion is expressed as,

$$\begin{aligned}
[\text{III} - 0] \left\{ \begin{aligned} \varphi_0 &= k \sum_{n=0}^{\infty} \frac{1}{(2n)!} \left(\int_0^{\infty} \lambda^{2n} (\psi_1(\lambda) \cosh \lambda b + \psi_2(\lambda) \sinh \lambda b) d\lambda \right) r^{2n} \cos 2n\theta \\ &+ k \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \left(\int_0^{\infty} \lambda^{2n+1} (\psi_1(\lambda) \sinh \lambda b + \psi_2(\lambda) \cosh \lambda b) d\lambda \right) r^{2n+1} \cos(2n+1)\theta \\ &= k \sum_{n=1}^{\infty} \alpha_{2n+1} r^{2n+1} \cos(2n+1)\theta + k \sum_{n=1}^{\infty} \alpha_{2n} r^{2n} \cos 2n\theta = k \sum_{n=2}^{\infty} \alpha_n r^n \cos n\theta, \end{aligned} \right. \quad (22)
\end{aligned}$$

$$\begin{aligned}
[\text{III} - 1] \left\{ \begin{aligned} \varphi_1 &= k \sum_{n=0}^{\infty} \frac{1}{(2n)!} \left(\int_0^{\infty} \lambda^{2n+1} (\psi_3(\lambda) \cosh \lambda b + \psi_4(\lambda) \sinh \lambda b) d\lambda \right) r^{2n} \cos 2n\theta \\ &+ k \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \left(\int_0^{\infty} \lambda^{2n+2} (\psi_3(\lambda) \sinh \lambda b + \psi_4(\lambda) \cosh \lambda b) d\lambda \right) r^{2n+1} \cos(2n+1)\theta \\ &= k \sum_{n=1}^{\infty} \beta_{2n-1} r^{2n} \cos 2n\theta + k \sum_{n=1}^{\infty} \beta_{2n} r^{2n+1} \cos(2n+1)\theta = k \sum_{n=1}^{\infty} \beta_n r^{n+1} \cos(n+1)\theta, \end{aligned} \right. \quad (23)
\end{aligned}$$

where

$$\begin{aligned}
\alpha_{2n} &= \frac{1}{(2n)!} \int_0^{\infty} \lambda^{2n} \{ \psi_1(\lambda) \cosh \lambda b + \psi_2(\lambda) \sinh \lambda b \} d\lambda \\
&= \sum_{m=1}^{\infty} [{}^m\text{I}_n A_{2m} + {}^m\text{III}_n B_{2m-1}] + \sum_{m=0}^{\infty} [{}^m\text{II}_n A_{2m+1}] + \sum_{m=1}^{\infty} [{}^m\text{IV}_n B_{2m}] \\
&= {}^0\text{II}_n A_1 + \sum_{m=1}^{\infty} [{}^m\text{I}_n A_{2m} + {}^m\text{II}_n A_{2m+1} + {}^m\text{III}_n B_{2m-1} + {}^m\text{IV}_n B_{2m}],
\end{aligned} \quad (24)$$

$$\begin{aligned}
\alpha_{2n+1} &= \frac{1}{(2n+1)!} \int_0^{\infty} \lambda^{2n+1} \{ \psi_1(\lambda) \sinh \lambda b + \psi_2(\lambda) \cosh \lambda b \} d\lambda \\
&= \sum_{m=1}^{\infty} [{}^m\text{I}'_n A_{2m} + {}^m\text{III}'_n B_{2m-1}] + \sum_{m=0}^{\infty} [{}^m\text{II}'_n A_{2m+1}] + \sum_{m=1}^{\infty} [{}^m\text{IV}'_n B_{2m}] \\
&= {}^0\text{II}'_n A_1 + \sum_{m=1}^{\infty} [{}^m\text{I}'_n A_{2m} + {}^m\text{II}'_n A_{2m+1} + {}^m\text{III}'_n B_{2m-1} + {}^m\text{IV}'_n B_{2m}],
\end{aligned} \quad (25)$$

$$\begin{aligned}\beta_{2n-1} &= \frac{1}{(2n)!} \int_0^\infty \lambda^{2n+1} \{\psi_3(\lambda) \cosh \lambda b + \psi_4(\lambda) \sinh \lambda b\} d\lambda \\ &= {}^0\text{VI}_n A_1 + \sum_{m=1}^\infty [{}^m\text{V}_n A_{2m} + {}^m\text{VI}_n A_{2m+1} + {}^m\text{VII}_n B_{2m-1} + {}^m\text{VIII}_n B_{2m}],\end{aligned}\quad (26)$$

$$\begin{aligned}\beta_{2n} &= \frac{1}{(2n+1)!} \int_0^\infty \lambda^{2n+2} \{\psi_3(\lambda) \sinh \lambda b + \psi_4(\lambda) \cosh \lambda b\} d\lambda \\ &= {}^0\text{VI}'_n A_1 + \sum_{m=1}^\infty [{}^m\text{V}'_n A_{2m} + {}^m\text{VI}'_n A_{2m+1} + {}^m\text{VII}'_n B_{2m-1} + {}^m\text{VIII}'_n B_{2m}].\end{aligned}\quad (27)$$

We can express the terms of right hand side of Eqs.(24)-(27) into

$$\begin{aligned}{}^m\text{I}_n &= \frac{1}{2(2m-1)!(2n)!} \int_0^\infty \frac{\lambda^{2m+2n-1}}{4\lambda^2 - \sinh^2 2\lambda} \\ &\quad \left[-1 + 4(\kappa - 2\lambda)\lambda + e^{-4\lambda} + 2(1+b)\lambda \cosh 2(1-b)\lambda + 2(1-b)\lambda \cosh 2(1+b)\lambda \right. \\ &\quad \left. - \kappa(\sinh 2(1+b)\lambda + \sinh 2(1-b)\lambda) \right] d\lambda,\end{aligned}\quad (28)$$

$$\begin{aligned}{}^m\text{II}_n &= -\frac{1}{(2m)!(2n)!} \int_0^\infty \frac{\lambda^{2(m+n)}}{4\lambda^2 - \sinh^2 2\lambda} \\ &\quad \left[2b\lambda(2\lambda + \cosh 2b\lambda \sinh 2\lambda) + (-2\lambda \cosh 2\lambda + \kappa \sinh 2\lambda) \sinh 2b\lambda \right] d\lambda, \\ &\quad \dots\dots\dots, \\ {}^m\text{VIII}'_n &= \frac{1}{2(2m)!(2n+1)!} \int_0^\infty \frac{\lambda^{2m+2n+1}}{4\lambda^2 - \sinh^2 2\lambda} \\ &\quad \left[1 + 4\lambda(\kappa + 2\lambda) - e^{-4\lambda} \right. \\ &\quad \left. - 2(1+b)\lambda \cosh 2(1-b)\lambda - 2(1-b)\lambda \cosh 2(1+b)\lambda \right. \\ &\quad \left. - \kappa(\sinh 2(1-b)\lambda + \sinh 2(1+b)\lambda) \right] d\lambda.\end{aligned}\quad (29)$$

Deriving the stresses from the potential functions given by [I], [II], [III-0] and [III-1] and satisfying the boundary conditions expressed by Eq.(4), we obtain two sets of equations for determining the unknown constants A_n and B_n .

(i) Normal stress around the hole must be free ($(\sigma_r/k)_{r=a} = 0$);

$$\begin{aligned}\left(\frac{\sigma_r}{k}\right)_{r=a} &= \left(\frac{2}{a^3}A_1 + \frac{2+\kappa}{a^3}B_1 - a\beta_1\right) \cos \theta + \left(\frac{6}{a^4}A_2 + \frac{3(\kappa+3)}{2a^4}B_2 + 2\alpha_2\right) \cos 2\theta \\ &\quad + \sum_{n=3}^\infty \left[\frac{n(n+1)}{a^{n+2}}A_n + \frac{(n-1)(n+2)}{2a^n}B_{n-2} + \frac{(n+1)(n+\kappa+1)}{2a^{n+2}}B_n \right. \\ &\quad \left. + n(n-1)a^{n-2}\alpha_n + \frac{1}{2}(n-1)(n-\kappa-1)a^{n-2}\beta_{n-2} + \frac{1}{2}(n-2)(n+1)a^n\beta_n \right] \cos n\theta \\ &= -\frac{a}{4} \cos \theta + \frac{a}{4} \cos 3\theta.\end{aligned}\quad (30)$$

(ii) Shear stress around the hole must be free ($(\tau_{r\theta}/k)_{r=a} = 0$);

$$\begin{aligned}\left(\frac{\tau_{r\theta}}{k}\right)_{r=a} &= \left(\frac{2}{a^3}A_1 + \frac{2+\kappa}{a^3}B_1 - a\beta_1\right) \sin \theta + \left(\frac{6}{a^4}A_2 + \frac{3(\kappa+3)}{2a^4}B_2 - 2\alpha_2 - 3a^2\beta_2\right) \sin 2\theta \\ &\quad + \sum_{n=3}^\infty \left[\frac{n(n+1)}{a^{n+2}}A_n + \frac{n(n-1)}{2a^n}B_{n-2} + \frac{(n+1)(n+\kappa+1)}{2a^{n+2}}B_n \right. \\ &\quad \left. - n(n-1)a^{n-2}\alpha_n - \frac{1}{2}(n-1)(n-\kappa-1)a^{n-2}\beta_{n-2} - \frac{1}{2}n(n+1)a^n\beta_n \right] \sin n\theta \\ &= -\frac{a}{4} \sin \theta - \frac{a}{4} \sin 3\theta.\end{aligned}\quad (31)$$

By equating the coefficients of $\cos n\theta$ and $\sin n\theta$ on the left- and right-hand sides of Eqs.(30) and (31), an infinite set of algebraic equations for the unknown constants A_n, B_n is obtained. Each set of unknowns is distinguished

by a subscript n , which varies from one to infinity. We evaluate these unknown constants by truncating the infinite number to $n = N$. The number N is set to 20 to satisfy the boundary conditions on the rim to at least three significant figures. All displacements and stresses are obtained by the linear combination of the potentials [I], [II] and [III].

3. Numerical Results

In the numerical example, we assume plane stress. Numerical calculations are carried out for various hole radii a , and the eccentricity b . Poisson's ratios ν is taken as 0.3 and hence $\kappa = 2.08$. The preceding infinite integrals are computed using Simpson's rule of numerical integration.

To confirm the validity of the present method, we compare our numerical results with those of FEM. In the FEM analysis, a triangular six node higher order element is employed and the strip is divided into about 8000 elements. Figure 2 shows the mesh pattern of FEM and von Mises stress contour obtained by FEM. Figure 3 shows the variation of hoop stress σ_θ/k around the hole for $a = 0.4$ and $b = 0.2$. In the figure, it can be seen that the present results agree well with the FEM ones. Figure 4 shows comparison of σ_θ/k for $a = 0.2, b = 0.2$ and $a = 0.6, b = 0.2$. For the larger hole ($a = 0.6$), it can be seen that the FEM results cannot be computed close to the present ones. This is mainly due to the discretization error. To illustrate the effects of the eccentricity b on stresses around the circular hole, numerical results are shown graphically in Fig.5 ~ Fig.7.

Figure 5 shows the relationship between stress σ_θ/k and angle θ (see Fig.1) for $a = 0.4$ and 0.6 . The curves in Fig.5 are no more inversely symmetrical with $\theta = 90^\circ$ and the value of stress $(\sigma_\theta)_{\theta=0}$ becomes larger than that of $(\sigma_\theta)_{\theta=180^\circ}$. Also, as expected, the eccentricity b has a small effects on the stress σ_θ around the hole when the radius a is small and a large effect when a is large.

Figures 6 shows the variations of σ_y along x -axis for $a = 0.2$ and 0.4 . For small hole ($a = 0.2$), it is seen that the curves tend to gradually approach to a straight line $\sigma_y = kx$ as $x \rightarrow 1$ and that stress concentration occur at the point A (see Fig. 1). On the other hand, the stress σ_y is more affected by the eccentricity b when for large hole ($a = 0.6$). For infinitesimally small hole in a elastic body under bending, it is well known the stress concentration factor at the point A is equals to 2. In order to show the influences of the eccentricity and the boundary of the strip on the stress concentration, stress $(\sigma_\theta)_{r=a, \theta=0}$ (is identical with $(\sigma_y)_{x=a}$) is considered.

Figure 7 shows the variations $(\sigma_\theta)_{r=a, \theta=0}$ with the hole radius a . When a is small, the variation is almost same as the straight line $(2a)$, which means the effect the eccentricity b and the edge boundary is little. But in the range $a > 0.4$, the rate of the stress concentration remarkably increases.

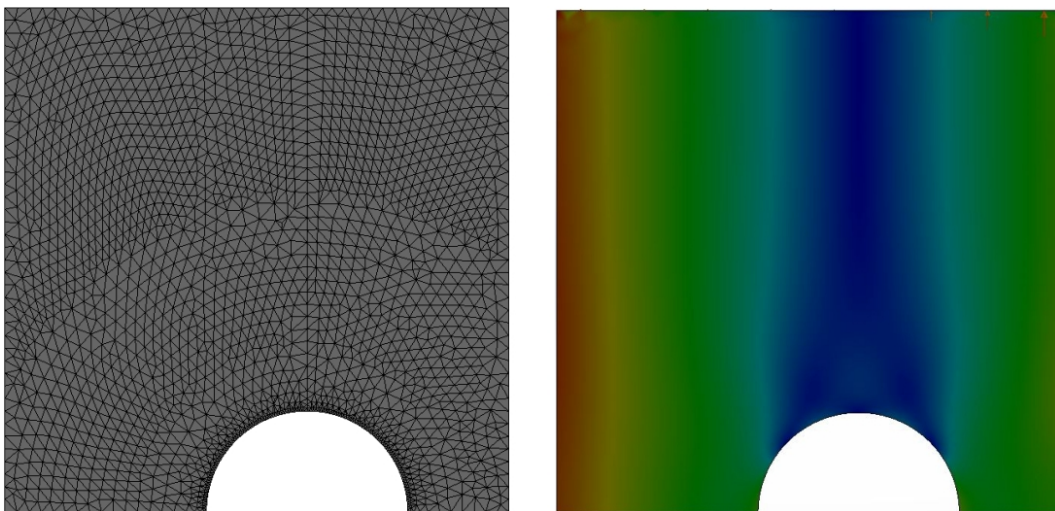


Fig. 2 Meshing example of the FEM and von Mises stress contour obtained by FEM analysis ($a = 0.4$, $b = 0.2$). Total number of elements in this example is 8537.

4. Conclusion

In this study we developed a method of solving asymmetric problem of an infinite strip having an eccentric

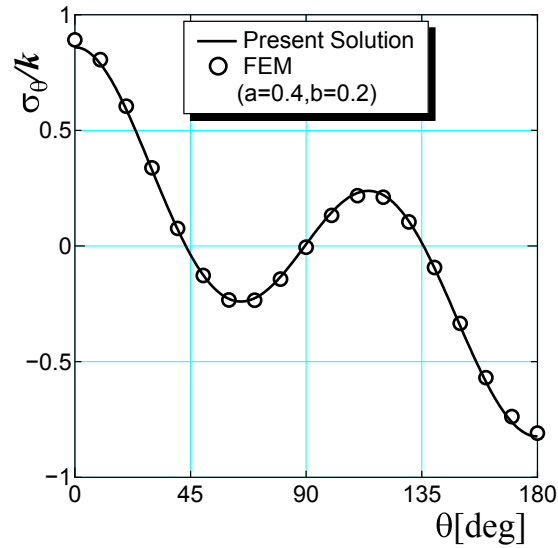


Fig. 3 Variations of σ_θ/k on the eccentric hole with θ for $a=0.4$, $b=0.2$. (solid line: present study, open circle: FEM result)

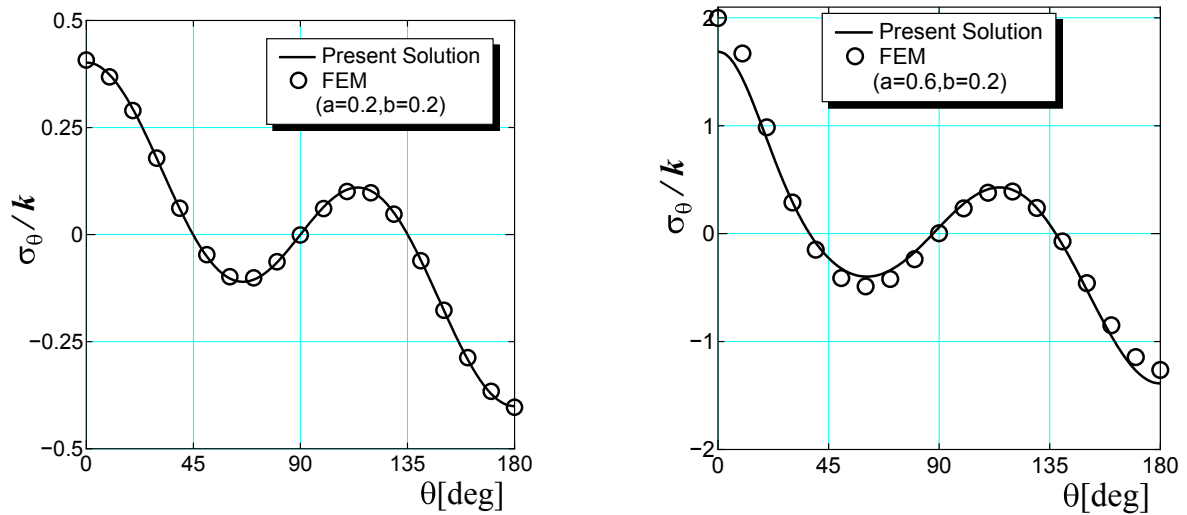


Fig. 4 Variations of σ_θ/k on the eccentric hole with θ for $a=0.2$, $b=0.2$ and $a=0.6$, $b=0.2$. (solid line: present study, open circle: FEM result)

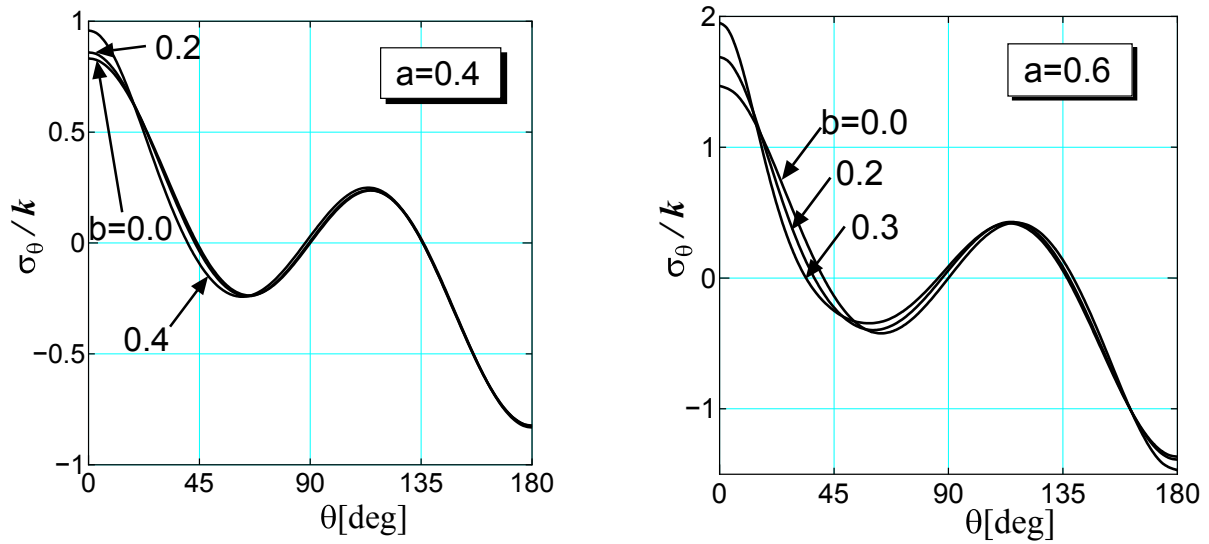


Fig. 5 Variations of σ_θ/k on the eccentric hole with θ for $a=0.4$ and $a=0.6$. The eccentricity b is taken as parameter.

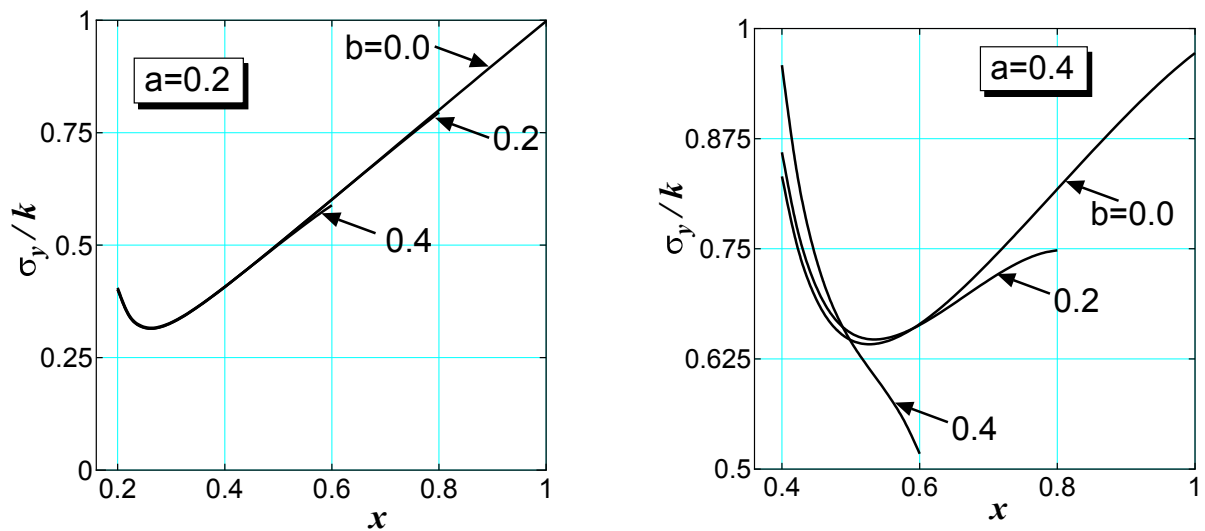


Fig. 6 Variations of σ_y/k along x -axis with x for $a=0.2$ and $a=0.4$. The hole radius a is taken as parameter.

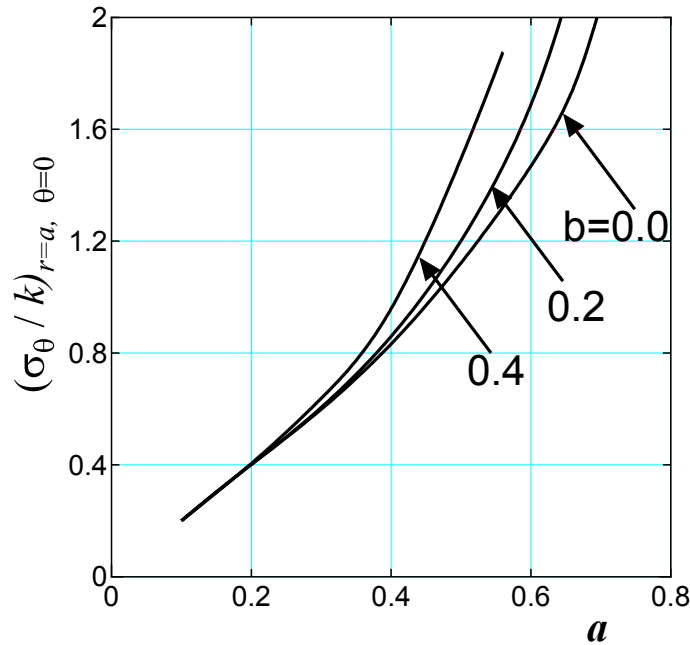


Fig. 7 Variation of stress $(\sigma_{\theta}/k)_{r=a, \theta=0}$ with hole radius a . The eccentricity b is taken as parameter. The stress $(\sigma_{\theta})_{r=a, \theta=0}$ is equivalent to the stress $(\sigma_y)_{x=a}$

circular hole when the strip was subjected to bending at infinity. Following an analytical procedure, we carried out some numerical calculations for the strip with an eccentric hole, showing the stress distributions around the hole and the effects of eccentricity b on stress concentration. On the basis of the preceding analysis, the following conclusions are drawn:

(1) The analytical solution for an infinite strip with an eccentric hole is represented in a closed form using the harmonic displacement potentials.

(2) The effect of the eccentricity b and the strip boundary are clarified through the numerical calculations.

The Papcovich-Neuber approach have an advantage over the Airy stress function one in the calculation of the displacements. In the future, using this advantage, we are going to analyze the stress field of the bending of an eccentric inclusion in a strip.

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